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Fig. 10.

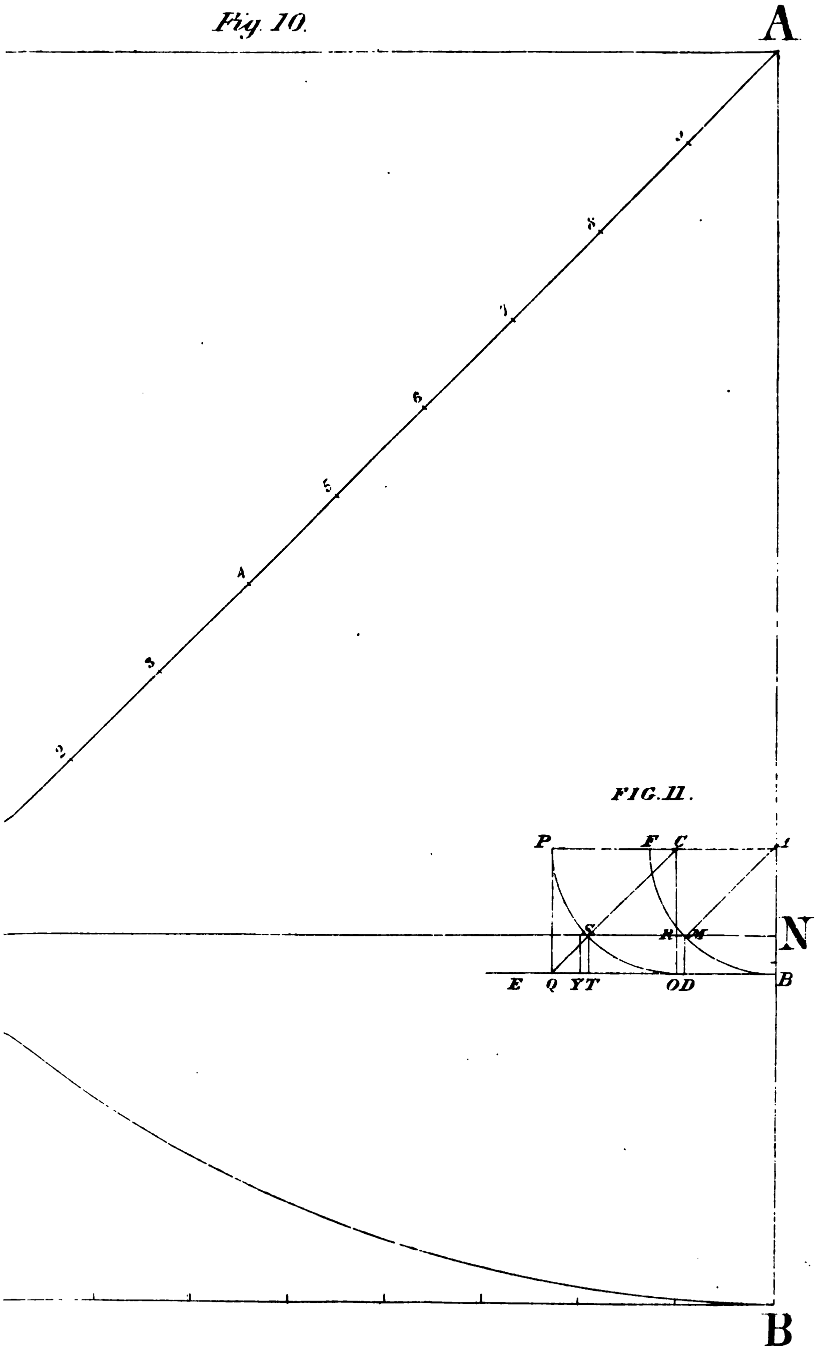
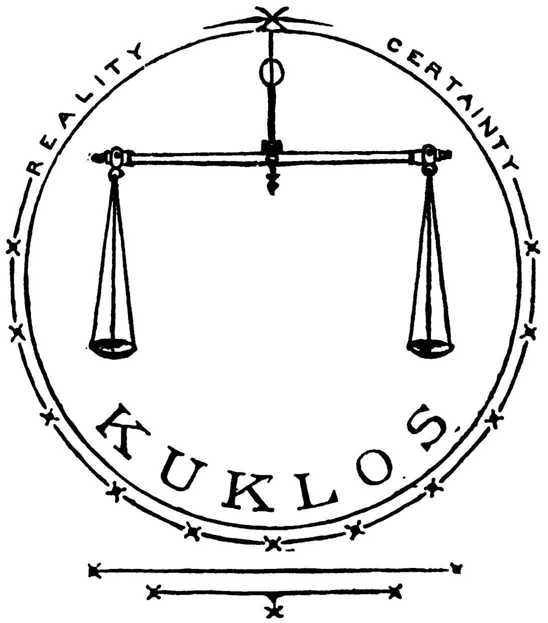


FIG. 11.

THE CIRCLE
AND
STRAIGHT LINE.



PART FIRST.

Entered according to Act of Parliament, in the year one thousand eight hundred and seventy-four, by JOHN HARRIS, in the office of the Minister of Agriculture and Statistics, at Ottawa.

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THE CIRCLE

AND

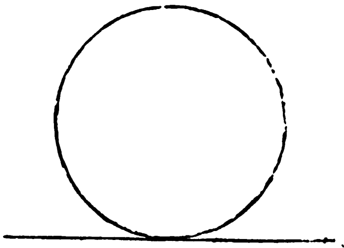
STRAIGHT LINE

PART FIRST.

THE CONSTRUCTION OF THE CIRCLE

IN RELATION TO

THE SCIENCE OF FORM AND MAGNITUDE.



BY

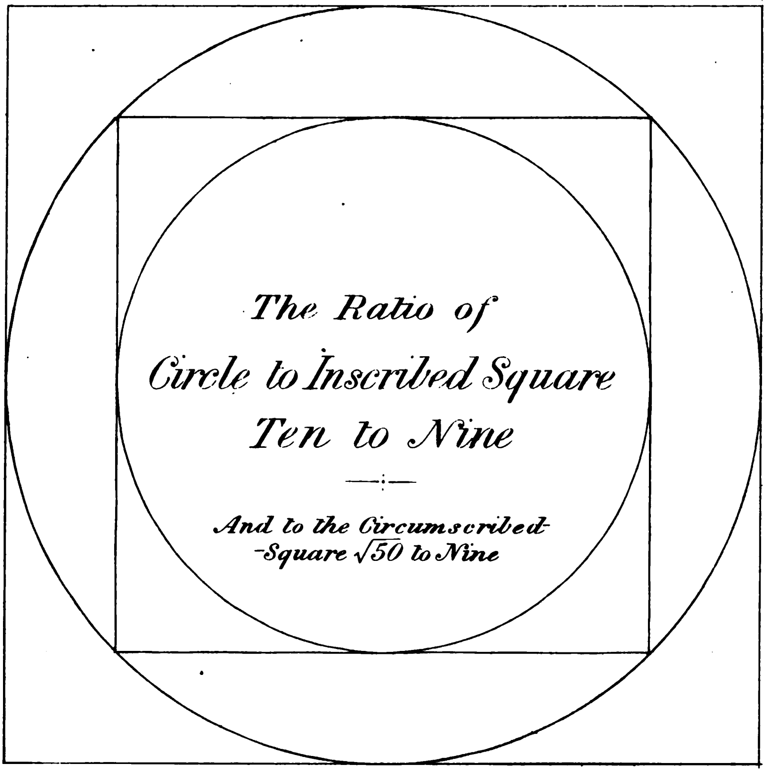
JOHN HARRIS.

MONTREAL:
JOHN LOVELL, ST. NICHOLAS STREET.

FEBRUARY, 1874.

Revised, Corrected and Reprinted,
September, 1875.

TO HIM
THAT GIVETH STRENGTH TO THE WEAK
AND
WISDOM TO THE SIMPLE,
TO WHOM
BELONGS WHATEVER MERIT IS CONTAINED IN IT,
THIS BOOK
IS
HUMBLY AND REVERENTLY
Dedicated.



The Ratio of
Circle to Inscribed Square
Ten to Nine

—:—

And to the Circumscribed-
-Square $\sqrt{50}$ to Nine

PREFACE.

DEEMING the subject of this work, viz: the solution of that geometrical problem which demonstrates the relation of the circle to the straight line, to be peculiarly of public importance, we will briefly state what has taken place in respect to it, and the present position of the matter.

The discovery of the solution was communicated by letter, dated 29th December, 1870, accompanied with demonstration, &c., to the Astronomer Royal. It was afterwards found that the case, as contained in the papers thus presented in the first instance, was in a strictly geometrical sense imperfect and faulty; the solution and demonstration, however, were virtually contained in them, and no objection whatever was made to their form.

The papers were acknowledged; but, after some considerable time, a decisive intimation was received from Greenwich that the Astronomer Royal declined to examine the case.

In the month of January, 1873, the papers were presented to the president of the Royal Society, with a formal request (claim) in writing to have the case judicially examined by that Society. The documents were returned with a note from the President, (G. B. Airy, Esq.,) decidedly refusing, as President, to receive the papers or to acknowledge the claim for investigation.

Before this refusal was received, considering that the subject if correctly understood would be found to have an especial interest for the public of a practical and immediate character, four letters containing a general explanation of the great importance of the subject with regard to the immediate interests of the public, were forwarded to the Editor of the "Times" (London) for publication in that Journal; and, when the letter of the President of the Royal Society, refusing to entertain the case, was received, a copy of that letter was also forwarded to the Editor of the " Times."

About the same time that the case was presented for investigation to the Royal Society, a communication thereof was made to McGill College, also formally requesting an investigation. Not long afterwards, however, the papers were returned, without, as it would seem, having undergone examination.

Copies of the several documents referred to will be found at the end of the Appendix to this book.

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These plates are in a separate cover, loose in the book, for convenience of reference.

INTRODUCTION.

The subject of this work is that geometrical problem which requires a straight line to be drawn equal in length to the given arc of a circle ; or, a circle to be described equal in length to a given straight line—accompanied, in either case, with demonstration that the conditions of the requisition have been mathematically fulfilled.

This problem we have succeeded in solving according to the strict rules of geometry, and the demonstration that we have so done, is herein presented to the public.

We publish our solution with the distinct statement that it is essentially in strict accordance with that scientific system known as Euclid's. We claim to have our demonstration admitted or disproved, and we challenge objection or adverse argument on that system.

The book known as 'Euclid's Elements of Geometry,' although possessing a high degree of completeness compared with other scientific treatises, and including a considerable part of the subjects belonging to that division of science, is a human production and imperfect—it is neither absolutely complete, nor absolutely comprehensive.

However desirable it may be to retain the formal method in and by which Euclid taught his application of the system, there can be no good reason why the method should be restricted to any particular number of problems, or why it should not be extended to include cases of the same character as those treated in the 'Elements,' but which did not come under the consideration of Euclid.

Let us briefly examine the essential characteristics of the scientific system taught by Euclid,—and also the arrangement of Euclid's formal method.

The first essential of the system is that there shall be a simple (elementary) basis . . . of which the reality, actuality, or truth, is absolutely certain. This fundamental basis (or each such basis, because there may be an indefinite number) being elementary and manifestly real and true, requires merely to be defined or stated with precision; and it is termed, accordingly,—*a definition.*

A geometrical definition may therefore be called the verbal equivalent of a fact. *

When the basis is not (elementary) quite simple, but requires definite statement, or brief explanation only, to satisfy the reasonable mind as to its unquestionable truth and reality . . it is called—*an axiom.*†

The postulates of 'Euclid's elements' are the rules of his systematic method, as distinguished from the laws or rules of the philosophy, which last are to be learnt from their application and illustration throughout the work. In stating this, we are not making a fanciful distinction, but indicating a difference which it is of great moment to correctly appreciate.

---

* It may be said to assert the existence of the fact which it defines.

† Strictly speaking an axiom is a proposition or theorem, of which the manifest truth becomes so readily apparent as to render formal demonstration unnecessary; or, in some instances, the definite statement of the axiom may include its demonstration. Therefore it is in the same case with the definition, and it may be considered a compound definition; or, the definition may be considered an elementary axiom. In either case there is the definition only of a reasonable recognition by the mind of an actual existence or reality. (Def.) *e. g.*, 'A straight line is that which lies évenly between its points,' the mind at once recognizes and *reasonably* accepts the *fact*, which is manifest whether the line be considered only a natural or only an ideal reality. (Axiom.) *e. g.*, 'The whole is greater than its part,' the comparison has already been made by the mind, and the inter-relation of the two things, thus defined, is at once recognized by the mind as actual, (*i.e.*, as an ideal fact.)

Euclid's 'Ideal Philosophy,' or 'Scientific System,' if perfected, would be a perfect system of human reasoning, that is to say, a system under which, its rules and regulations being strictly observed, the guidance of reason could be obtained (in compounding knowledge) on all subjects to which human knowledge is permitted to extend, so as to insure the attainment of certainty and truth in all cases.

Euclid's systematic method, as taught in the work known as the 'Elements of Geometry,' is an endeavour to apply that system to one of the divisions of science, — namely, the science of Form and Magnitude; an endeavour which, with the exception of one grievous and calamitous mistake, must be considered as having been successful in an astonishing degree; for, as a complete, comprehensive and important work in itself, the (so-called) 'Elements of Geometry' may be justly regarded as the greatest product, and as the proudest monument, of the human intellect the world has yet to show.

The postulates of the 'Elements' are the arbitrary rules * framed to insure method and precision in the application of the system to the science of 'Form and Mag-

---

* Arbitrary, however, in a relative, not in an absolute sense; for the postulates of 'Euclid's Elements' are reasonable, *i. e.*, subordinate to the rules of reason. The postulates are arbitrary in a general sense, in the same relative manner that the laws of material Nature are arbitrary in a spiritual sense; in either case there is a limitation and restriction which may be termed artificial, but . . the one is a contrivance of human reason . . the other, an arrangement by Divine reason.

nitude.' * Formally, they give permission to conduct the necessary process of construction in a particular manner; thereby strictly prohibiting other methods of conducting the process. Now the importance of such strict rules in science generally, and in each of the divisions of science, is not to be lightly esteemed. For the purpose of preserving order and distinctness, and of enabling a number of persons to work harmoniously (in concert) on the same subject, some such rules may be considered practically indispensable; the indirect benefit of their influence, also, in training the mind to a due regard for law and system, in its endeavours to acquire increased knowledge by compounding the elements thereof, is unquestionably very great.

It would be, however, superstitious, slavish, and quite unreasonable, for men to allow themselves collectively or individually to be absolutely bound and fettered by a particular set of rules, framed long since by other men, viz., by Euclid and his predecessors, as the most advan-

---

* The science of 'Form and Magnitude' may be correctly termed an abstract science, because the expression 'abstract is so used in a definite and intelligible sense. To us as human beings, cognizing even our ideas and thoughts through a material medium, the abstract property...such as form, magnitude, number, or quantity, is not conceivable as an existence distinct from a material existence or thing, the same as a spiritual existence or force is conceivable as distinct. The abstract sciences belong, therefore, to natural science, (*i. e.*) to the science of Material Nature, which includes:—

The material sciences.

The sciences of spiritual force manifested on matter.

The science of the abstract properties belonging to matter.

By the aid of ideal philosophy, we are enabled to separate and to study separately these...all and each of which form a part of compound matter, (*i. e.* of matter naturally recognized as such by us.)

tageous which could at that time be devised. On the contrary, if circumstances render an extension or modification of the rules desirable, it is quite reasonable, deliberately and circumspectly, to make and agree to such alteration as the advancement of science may call for.

The formal method adopted and taught by Euclid may be described as consisting of:—(1) The definite statement of the requisition. (2) The construction and arrangement of the case, (*i.e.*, of the circumstances and facts belonging to the case). (3) The answer to the requisition. (4) Demonstration that the answer is the true one. In this manner, the legitimate process of compounding knowledge is, by the aid of reason, systematically and methodically applied to the particular facts of the science; and which science in Euclid's treatise is that of Form and Magnitude.

The proposition or theorem is first stated in precise but general terms. A particular case, suitable for the illustration of the question, is then taken, and by aid of the geometrical figure is distinctly defined. Next comes the Construction. The Construction may be considered as equivalent to an experiment in physical or chemical science*; certain additional lines are drawn for the purpose of giving completeness to the figure, so as to enable the necessary combinations and comparisons to be made in an orderly and systematic manner. The geometrician is now in a position to obtain the answer to the requisition. This is at first briefly stated as a positive or decided conclusion. The demonstration then follows; in which the answer to the requisition, as stated, is justified by the facts upon which it is based, and shown thereby to be the reasonable and only correct explanation of the result.

---

* The construction in Euclid's method is the orderly mode of submitting the solution of the problem for the approval of reason, . . . it is in the first place a result, ... the mind has arranged the elements of the case and contemplates the conclusion resulting therefrom; but, before definitively accepting the conclusion, formally submits the arranged case to reason for approval. If demonstration follows, reason approves; if reason disapproves, demonstration fails. Therefore, 'Euclid's Construction' is equivalent to the (so-called) experiment in physical or chemical science: the experiment is, also, in the first place, a result; . . . the conditions of the case are arranged by the mind which contemplates the conclusion, but which, before venturing to accept that conclusion, submits the arranged case (together with

We have elsewhere stated that the primary and most important of the two-fold purpose, which Euclid's work had in view, was to teach and illustrate the philosophy—*i.e.*, the scientific system of reasoning, and that the application of the philosophy in his treatise on the science of 'Form and Magnitude,' does not justify the inference that the philosophy has a peculiar connection with that one division of science, and that it is not, with the requisite modifications, equally applicable to the other divisions of science.

* *Note* (*a*). In writing thus it is to be understood that we are idealizing Euclid, and considering him as the representative author of a work which cannot reasonably be supposed the production of one individual only;

---

the conclusion that it contemplates accepting) for the approval of reason. If the experiment (properly performed) succeeds, reason approves; if reason disapproves, the experiment fails. It may be an exhibition experiment: the experimenter has himself accepted the result (he has repeatedly performed the experiment and has knowledge that reason approves the arranged case), but he exhibits the experiment. For what purpose? In order to demonstrate to the spectators the approval of reason; or, more strictly speaking, to demonstrate the legitimacy of the arrangement and soundness of its conclusion, which conclusion is thus authorized and commended by reason for the mind's acceptance as knowledge.

it may be that a large part of the 'Elements,' as well as the arrangement of the parts and the coherent completeness of the exposition as a whole, is attributable to Euclid himself, but to some extent, at least, the book must be considered as the arrangement of the work of his predecessors by Euclid.—

We do not assert that Euclid himself had, in arranging the work, a clear and distinct appreciation of the philosophy as a system of reasoning not peculiarly connected with that one division of science to which it is applied in the 'Elements.' It is almost certain that he had not, and it is, humanly speaking, almost impossible that he could have had such a distinct appreciation of the fact. In the first place, the choice of the subject of the application was not originally his, the compound of the philosophy with that particular division of science was known as Geometry before his time, and was already regarded, not as a compound, but as a peculiar (mystical) division of knowledge; if Euclid had clearly perceived the fallacy of thus regarding it, some decisive indication that he did so would appear in the work. And, secondly, it must be remembered that most of the divisions of science, as we now recognize them, were at that time quite unknown, and, it may be said that, scientific knowledge was in a great measure comprised in that one division which alone had been scientifically arranged. It was, therefore, evidently almost impossible for Euclid to clearly appreciate the general relationship of philosophy to all the divisions of science, because he was quite unacquainted with even the existence of the major part of those divisions, and was consequently, not in a position to appreciate the inter-relation of the divisions to each other as parts of one general science.

* *Note* (*b*.)—There is however some apparent evidence that Euclid did in a measure regard the exposition of the particular science *primarily* as a means of illustrating and

teaching the general philosophy : instances of propositions treated in a very indirect and elaborate manner, at least, suggest such an explanation. Notable instances of this kind are the 1st prop. of Book III, and the 2nd prop. of Book XII.*—

The rules of philosophy are not in the same case with the postulates of Euclid's formal method. The laws of reason—that is, the authoritative rules by which knowledge is to be compounded, and according to which compound knowledge is to be accepted as sound, or rejected as unsound, cannot be altered, and may not be tampered with in any degree, nor is it permissible to disregard them. An unintentional neglect of these rules through ignorance or carelessness entails in all cases some degree of retributive punishment on the offender, according to the circumstances and the greater or lesser importance of the subject on which the mind thus neglects the laws of its intellectual existence. A wilful disregard, contempt, or defiance, of these rules is an intellectual crime, and, it may be, if the subject be of great importance, and, especially, if the mind be of great capacity and the acknowledged representative of many others, a crime of frightful magnitude, of which the consequences fall, not only on the individual offender, but on an indefinite number (a vast number, perhaps) of other persons.

It is our purpose to bring, almost immediately, this subject, namely, the 'law of intellectual existence' and the 'responsibilities belonging to knowledge,' particularly before the public.

We will now proceed to the demonstration of the geometrical problem, in its several forms—that is to say, the several forms of the one general relationship in form and magnitude between the straight-line and the perimeter of the circle.

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* We will in the Appendix make some examination of these propositions relatively to the above suggestion.

THE CIRCLE AND STRAIGHT LINE.

PART FIRST.

1. *Definition*—If a circle be applied upon a straight line, and the circle be then moved upon the straight line in such wise that each point in the circumference, successive to the point in the circumference which is first in contact with the straight line, be brought successively into contact with the straight line—to wit, with each similar successive point in the straight line, commencing from the point in the straight line which is first in contact with the circle, and if the circle be in such wise continually moved until a given point in the circumference, at a definite distance from the first point of contact, become in contact with the straight line—the circle is said to be rolled upon the straight line from the point in the circumference first in contact with the straight line to the given point in the circumference.

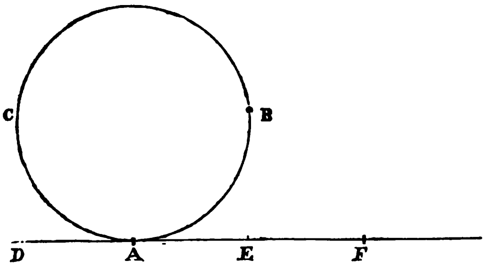


Fig. 1.

(Fig. 1.) Let *A.B.C.* be the circle applied upon the straight line *D.E.F.*, and in contact with the straight line at *A.*, and let *B.* be the given point in the circumference. If the circle be moved upon the straight line in the direction *D.E.F.*, and be so moved that each successive point in the circumference between *A.* and *B.* become successively in contact with each similar point successive in the straight line from the point *A.*, and be so moved continually until the given point *B.* in the circumference of the circle be in contact with the straight line—the circle *A.B.C.* is said to be rolled upon the straight line *D.E.F.*, from *A.* to *B.* on the circle. (*a.*)

But if the given point be upon the straight line, at a definite distance from the first point of contact between the circle and the straight line, and the circle be moved, in such wise as before, until some point in the circumference become in contact with the given point on the straight line—the circle is said to be rolled upon the straight line from the first point of contact to the given point on the straight line.

(Fig. 1.) Let F. be the given point on the straight line, and let the circle *A.B.C.* be continually moved from the point *A.*, in the direction *D.E.F.*, in the manner directed, until some point in the circumference become in contact with the point *F.* on the line—the circle *A.B.C.* is said to be rolled upon the straight line from *A.* to *F.* on the line. (*b.*)

2. *Definition*—If a straight line be made to deviate from its evenness between the two extreme points thereof in such wise that the line be made to contain a circle or to contain any part of the circumference of a circle—the straight line is said to be bent or curved into the arc of a circle.

Similarly if a curved line be extended until it lies evenly between its extreme points—the line is said to be unbent into a straight line.

3. *Definition*.. Arcs which are similar fractions of circles of different magnitudes are called similar arcs of different magnitudes. *e. g.*, if *A. B.* be an octant belonging to a circle of any definite magnitude, and *C. D.* be an octant belonging to a circle of twice the magnitude of the first; then *C. D.* is called an octant or similar arc of twice the magnitude of *A. B.*

Postulates.—Let it be granted that:

1. A circle, or that any arc of a circle may be rolled upon a straight line.

2. A straight line may be bent into a curved line.

3. The arc of a circle, or other curved line may be unbent (extended) into a straight line.*

Axioms.—1. The arc of a circle formed by bending a straight line into a curved line is equal in length to that straight line.

2. The straight line formed by unbending the arc of a circle into a straight line is equal in length to that arc.

3. If a circle be rolled upon a straight line from the point of contact in that straight line until a given point in the circumference become in contact with the straight line . . . the distance between the first point of contact and the last point of contact on the straight line, is equal to the distance between the first point of contact and the given point on the circumference of the circle.

4. If similar arcs of different magnitudes be rolled upon a straight line from the original to the terminal extremity of each arc, the distances on the straight line between the first and last points of contact are respectively as the differences in the magnitudes of the similar arcs. (Because the ratios of the circumferences of circles are as the ratios of their radii.)

---

*Should any one be disposed to object that the operations for the performance of which leave is here taken are of a mechanical nature, a little consideration may convince them that describing a circle with a centre and radius is quite as much so. But in fact the circle is to be geometrically not mechanically rolled, and the reasoning is quite independent of any mechanical operation in both cases. It may be thus explained :—The circle is merely supposed to be rolled. The reasoning investigates and determines the alteration which would be occasioned in the relative positions of certain points if the circle were to be rolled.

Verbal definition.—When a magnitude is said to contain a certain number of lesser magnitudes, it is defined that the greater magnitude is wholly compounded of that certain number of those lesser magnitudes; similarly, when a line (or a definite quantity of longitudinal space) is stated to contain a certain number of (equal) parts, the line (or definite quantity of space), is defined by the statement to be wholly compounded of that certain number of those parts.

FUNDAMENTAL PROPOSITION.

That the difference between the sine-length and arc-length of the Octant or (between the chord-length and arc-length of the Quadrant), is one-tenth of that arc-length.

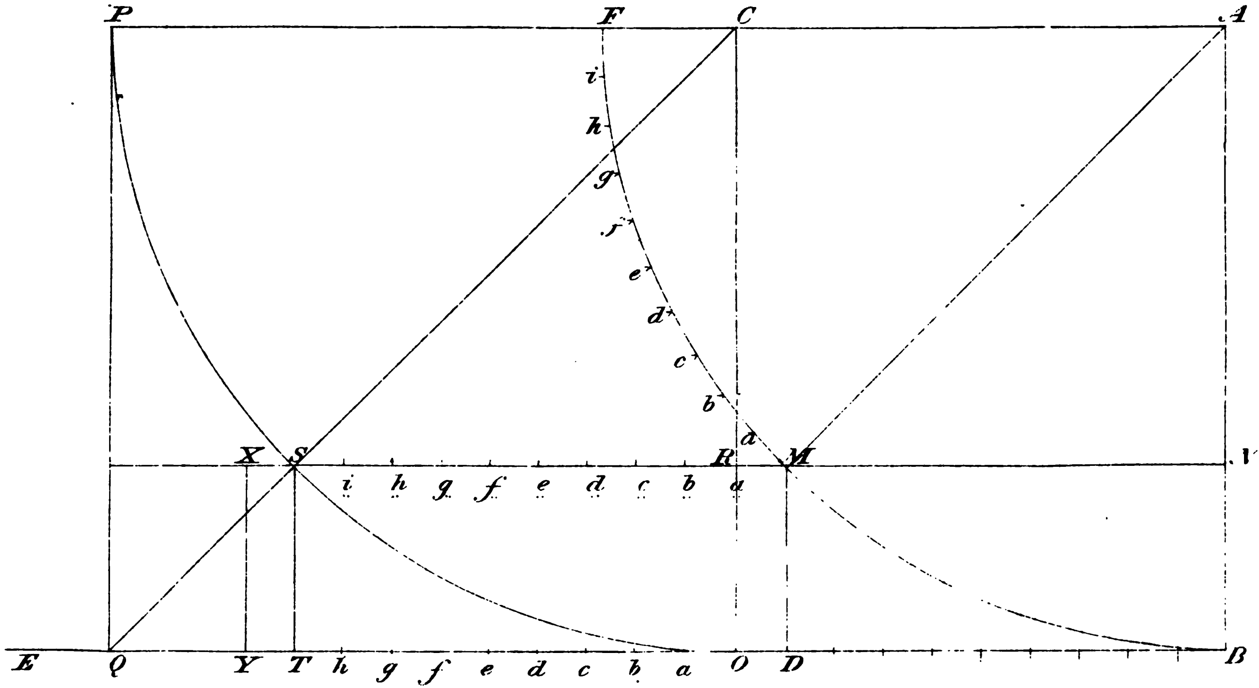
CONSTRUCTION ; FIGS. 1 & 2.

With the centre *A.* and radius *A.B.* describe the quadrant *B. F.* Bisect *B. F.* at *M.* Draw *M. N.*, the sine of the arc *B. M.*, at right angles to *A. B.*, intercepting *A. B.* at *N.* From *B.* at right angles to *A. B.*, draw *B. E.* of indefinite length, tangential to the arc *B.M.* Divide the arc *M.F.* into ten equal parts at the points of equal division *a.b.c.d.e.f.g.h.i.* From *M.* draw *M. D.* perpendicular to *B.E.* and intercepting *B. E.* at *D.*

Scholium.—If the curved line, forming the arc *B. M.* cut off from the quadrant at *M.*, is supposed to be straightened upon the line *B. E.*, the line *B. M.* will then throughout its entire length be applied upon the line *B. E.*, (*i.e.*, will coincide with a part of the straight line *B. E.*,) and the point *M.* at the extremity of *B. M.*, will manifestly be in contact with (*i.e.* coincide with) a point in the line *B. E.* at some place more distant than the point *D.* from *B.* Let *O.* indicate the point of contact of *M.*, when the curved line *B. M.* is straightened upon the line *B. E.*

Again, if the arc *B. M.*, in contact with the line *B. E.* at *B.*, is supposed to be rolled from *B.* upon the line *B. E.* until the point *M.* at the opposite extremity of the arc become in contact with the line *B. E.*, then, since each and every constituent (or component) part of the arc *B. M.* is successively brought into contact with each similar constituent (or component) part of the line *B. E.*, the

FIG. 1.



point of contact must necessarily be the same point *O.* touched by *M.* when the curved line forming the arc *B. M.* is straightened upon *B.E.*, and the straight line *B. O.* (*i.e.,* the straight line contained between the point *B.* and the point of contact indicated by *O.*) must necessarily contain the same quantity of length contained in the arc *B.M.*

Construction continued.—Suppose the arc *B.M.* to be straightened upon the line *B. E.*; and let *O.* indicate the point on the line *B. E.* which coincides with the point *M.* at the extremity of *B. M.* when so straightened. From *O.* perpendicular to *B. E.* draw *O. C.* equal in length to *A. B.* With centre *C.* and radius *C. O.* describe the quadrant *O P*, Bisect *O. P.* at *S.* Through *C. O.* join *S. M.* intersecting *C. O.* at *R.* and through *S.* draw *C. Q.* intercepting *B. E.* at *Q.* Join *C. P.*, *P. Q.*

From *S.* perpendicular to *B. E.* draw *S. P.* Produce *M. S.* through *S.* intercepting *P. Q.*; and, from the point *X.* on the line so produced, taken at the distance *S. X.* equal to the distance *M. R.* on the same line, draw the perpendicular *X. Y.*, intercepting *B. E.* at *Y.* Divide the line *R. S.* into nine equal parts at the points of equal division *b̤. c̤. d̤. ḛ. f̤. g̤. h̤. i̤.*

Preliminary Examination of the Construction.

Since, if the quadrant *B. F.* be rolled upon the line *B E.* until the point *M.*, bisecting the quadrant, arrive at *O.* (which indicates the unknown actual point of contact), the point *F.* at the extremity of the quadrant, must necessarily arrive at *S*, (*O. S.* represents what will then be the place of *M. F.*, because *M.* arrives at the point indicated by *O.*, and *F.* then, necessarily, arrives at *S.*), the point *e.* bisecting the arc *M. F.* must also, necessarily (when the arc *B. M* has rolled through half its length) arrive at *e.* on the line *M. S*, because the line *M. S.* is equal to the arc *M. F.*; it, therefore, follows that each of the divisional points *a. b. c. d. e. f. g. h. i.* on the arc *M. F.* must arrive successively at each of the divisional points *a̤. b̤. c̤. d̤. ḛ. f̤. g̤. h̤. i̤.* on the line *M. S.* Again, if the quadrant be further rolled upon the line *B.E.* until the point *F.*, at, the extremity of the quadrant, be in contact with *B. E.*, the point of contact must necessarily be the point *Y.* (because the arc *O. S.* (or *M.F.* is similar and equal to the arc *B. M.* and the movement of the one arc is similar and equal to the movement of the other) and *T. Y.* is equal to *D. O.*, therefore, since *B.* arrives at the point indicated by *O.*, and since *F.* (or *S.*) must then necessarily (if the rolling be continued)arrive at *Y.*, the line *O. Y.* is equal to the line *M. S.*, which is equal to the line *B. O.*, which by the hypothesis of the construction is the same length as the arc *B.M.*

Scholium.—By the Proposition upon which the construction is based, the first divisional point *a.* on the line *M. S.* falls on the point *R.*, and the space *M. a.* equals the space *M.R.*

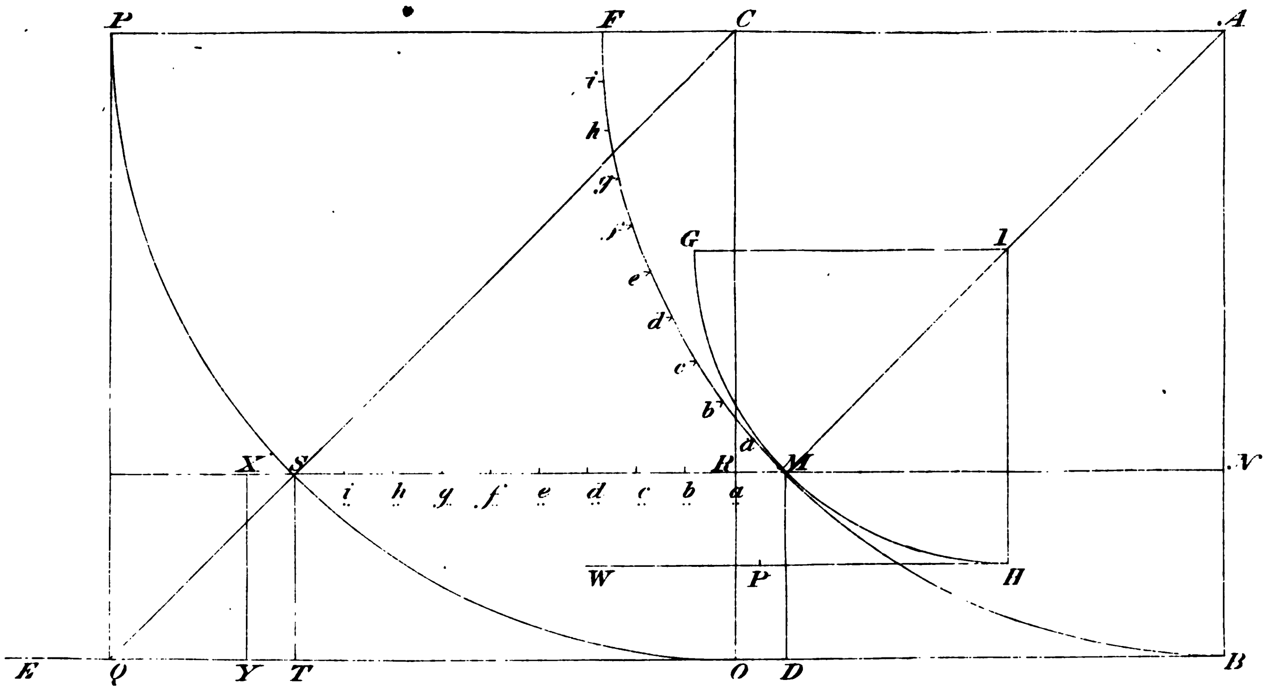
We have to demonstrate that the point on the line *M. S.* does coincide with the point *R.*—

Does the construction furnish a basis for geometrical demonstration ?

Divide the radius *A.M.*, of the arc *B.M.*, at *I.*, the point of bi-section. With centre *I.* and radius *I.M.*, describe the quadrant *H. G.*, half the length (magnitude) of the quadrant *B.M.P.* (because the radius *I.M.* is half of *A.M.*) The point *M.* which bisects the greater, bisects also the lesser quadrant; and the arc *M.C.*, the half of the lesser, is similar to *M.F.*, the half of the greater quadrant. From the point *H.* draw the tangential line, of indefinite length, *H.W.*, and bisect the space on that line contained between the perpendiculars *R.O.* and *M.D.*, in the point *P.*

Now, let the arc *H.M.G.* be rolled upon the straight line *H.W.* until the point *M.* become in contact on the line. . . Since the arc is similar to the greater arc and the motion is similar, and the lesser contains one-half the length of the greater, it is manifest that the point *G.*, at the extremity of the arc will intersect the line *M.S.* at point of bisection of that line . . to wit, at the point *e.*, half the distance of *S.* from *M.*, on the line *M.S.* But

FIG. 2



since the two arcs are similar, and the magnitude of the lesser is one-half of the greater, the sine of the lesser is also one-half the sine of the greater. Consequently, the remaining half *e.S.* of the line *M.S*, must contain one-half of the sine-length and one-half of *R.M.* the difference between the sine and arc-length of the greater arc. If, therefore, the sine *S.R.* of the greater arc be evenly divided, *e.S.* contains one-half of those even divisions, together with one-half of *R.M.* But *e.S.* also contains one-half of the even divisions contained in the line *M.S.* (arc-length). Evidently, therefore, each of the nine equal parts in the sine *R.S.* is compounded of a divisonal part of that line together with a similar divisional part of *R.M.* Also, it is evident that if the entire line *M.S.* (which is the arc-length) be evenly divided into ten equal parts, each of the divisional points must correspond with each similar point in the arc, and, in that case, if the greater quadrant be rolled through the half octant on the line *B.E.*, the central point *e.* in the octant *M. F.* must intersect the point *e.* on the line *M.S.* But can it be shown that the eight divisional points dividing the sine *S.R.* into nine equal parts, must necessarily coincide and be identical with eight of the divisional points dividing the entire line *S.M.* into ten equal parts, if that line be so evenly divided?

NOTE.—The foregoing examination has for its purpose to facilitate a preliminary apprehension by the student of the fundamental facts upon which the following investigation is based. In reference to Euclid's Elements, its purpose and character may be thus explained. The first proposition, for instance, in that work, commences with directions to construct a certain figure. The directions are quite definite and subjective; certainly they are not the random directions to draw an experimental figure which might or might not serve the intended purpose. It is evident therefore, that antecedent knowledge, an empirical and general knowledge, of the particular subject to which the construction belongs, must have been in the mind of Euclid, to enable him thus to present the case in such a form as to necessitate the reasonable acceptance of his conclusion, in other words, he must have had antecedent reasons for directing the particular construction

Let it be supposed that the divisional part *R.M.* is unequal to each of the other divisional parts, and let it be less than each of the nine equal divisions of *S.R.* Now, if the lesser quadrant *H.M.G.* be rolled the half distance as before, the point *G.*, at the extremity of the arc will necessarily intersect the line *M.S.* at the point of bisection, and the half of that line so cut off must contain half the sine *S.R.* together with half of the divisional space *R.M.*; but by the supposition *R.M.* is less than one of the nine equal divisions of *M.R.*, and, therefore, the point *G.*, at the extremity of the arc, cannot, by the supposition, intersect the line *M.S.* at the point *e.*, if *e.* be one of the divisional points dividing the sine *S.R.* into nine equal parts, because the half of *M. S.* cut off by the point *G.* contains one-half of *S.R.* together with one-half of *R.M.*, which (*i.e.*, *R.M.*) by the supposition is less than one of the equal parts of *S.R.* If the supposition were sound in fact, evidently, the sine *S.R.* could not be, then, evenly divided (into nine equal parts) because, if the entire arc-length be evenly divided (into ten equal parts), each part is compounded of a part of *S.R.* and a part of *R.M.* in the same ratio as the ratio of the whole of *S. R.* to the whole of *R.M.* Therefore, if *S.M.* be divided into ten equal parts, and *S.R.* be divided into ten equal parts, each one of nine of the equal divisions of *S.R.* will compound with one-ninth of *R.M.*, consequently the remaining one-tenth of *R.M.*

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to be drawn. The purpose of the above examination is to put the student in possession of our similar antecedent reasons for dictating the constructions belonging to the following propositions. A legitimate accordance with Euclid's method alone, would not necessitate our furnishing any such preliminary examination. It would be sufficient to commence at once with our first theorem in which the demonstration is quite independent of the preliminary examination. Mindful however, that we are here engaged in laying the foundation of Cyclometry as a very important subdivision of legitimate science, we feel assured the method adopted will be found more advantageous.

must likewise compound with one-tenth of *S.R.*, because all parts of the arc-length are similarly compounded. It follows, therefore, that the supposition is unsound, and, that the point *G.* at the extremity of the lesser arc, which bisects the line *M.S.* must necessarily intersect the point *e.* of equal division in that line; and it becomes apparent that if the line *S.R.* be evenly divided into ten equal parts, and to each of the equal parts, one-tenth part of *R.M.* be added, the ten parts will be increased into ten equal divisional parts of the line *S.M.*, of which ten equal parts the line *S.R.* will contain nine.

Hence it is evident that the construction affords the means of demonstrating the proposition.

THE FUNDAMENTAL PROPOSITION.

Theorem.—*If the octant of a circle be divided into ten equal parts, the sine of the octant shall contain nine equal parts, each of them equal to each of the equal parts contained in the octant.*

DEMONSTRATION, by the construction. Fig. 1 (already described)

Since, if the quadrant *M. F.* be rolled upon the line *B. E.* until the point *M.*, bisecting the quadrant, arrive at the point of contact on the line *B. E.* that place of contact must be a point in the line *B. E.* more distant than the point *D.* from the extremity of the line *B. E.* Let the place of the point of contact be tentatively indicated by *O.*, (leaving the actual distance of the point *O.* from the point *D.*, to be determined by demonstration.) Now, whatever may be the correct distance of the point *O.* from the point *D.*, it is evident that when *M.* arrives at the point of contact, the point *F.*, at the extremity of the quadrant, must necessarily arrive at *S.*, because *O. S.* represents what will then be the place of the octant *M. E.* It is also evident that, wherever the precise place of contact on the line *B. E.* may be, a perpendicular drawn from that place (to wit, the place indicated by *O.*) must be the line *O. R. C.* which intersects the line *M. S* in the point *a.*, because the point *M.* (which arrives at *O.*) is the extremity of the octant *M. F.* and the point *a.* is the extremity of the sine of that octant.

The octant *M. F.* is divided into ten equal parts at the points of equal division, *a. b. c. d. e. f. g. h. i.* It is evident that if the quadrant be rolled until the first divisional point *a.* becomes in contact on the line *B. E.* the distance of that point of contact *a.* from the point of contact *O.* must be the same as the distance between the point *M.* and the point *a.* in the octant *M. F.*, for *O.*, the point of contact on the line *B.E.*, represents the point *M.* when it arrives at the line *B. E.*

Now, suppose the point *M.* to be in contact on the line *B.E.*, and let the quadrant be rolled back again until the extremity of the quadrant be again in contact at *B.*, the extremity of the line *B.E.* It is evident that since the curvature of the arc, which is a part of the circle, is uniform, the motion of each of the points in the arc in moving downwards from the line *M.S.* to the line *B.E.*, when the quadrant rolls forwards, and also in moving upwards from the line *M. S.* to the line *B.E.*, when the quadrant rolls backwards, must be similar and equal to the motion of each of the other divisional points on the arc. Consequently the motion of the point *a.* is similar and equal to the motion of the point *M.*, and, therefore, since *M.* returns from the point *O.* and moves towards *N.*, through the horizontal distance *O.D.*, which is the same as the distance *R.M.*, the point *a.* moving from the point of contact *a.*, on the line *B.E.*, must move through an equal space and must intersect the line *M.S.* in the point *a.*, at the same distance *M.a.* from the point *M.* on the line *M.S.*, as the distance *M. a.*, on the octant *M.F.* and as the distance *M.a.* on the line *B.E.* Again, let the quadrant be rolled until the second divisional point *b.* of the octant *M.F.* become in contact on the line *B.E.* Since when the point *b.* passes through the line *M.S.*, the arc must intersect that line similarly, and at the same angle of intersection, as in each case respectively when each of the points *a.* and *M.* pass through that line, it is evident that the distance of the point of intersection *b.*, on that line, from the point of intersection *a.*, on that line, must be the same as the distance of the point *a.* from the point *M.*, because the motion of the point *b.*, when the quadrant is rolled upon the line *B.E.*, is equal and similar to the motion of the point *a.*, and to the motion of the point *M.* It is also evident that when *b.* arrives at the point of contact on the line *B.E.*, the distance of that point from the point *a.* will be the same as the distance of the point *a.* from *O.*, and the same as the distance of the point *b.* from the point *a.* on the octant *M. F.* Now, since the point *M.* moves from the point of contact *O.*, on the line *B. E.*, intercepted by the perpendicular *R. O.* (which intercepts the extremity of the sine *S. R.*), to the place *M.* on the line *M. S.*; and since the point *a.* moves from the point of contact on the line *B. E.*, to the point *a.* intersecting the line *M. S.* at the extremity of the sine *S. R.*, and since, when *M.* is in contact on the line *B. E.* at *O.*, the point *F.* at the extremity of the quadrant necessarily arrives at *S.*, it follows that each of the successive divisional points in the octant *M. F.* from the point *a.* to the point *F.* must pass through each successive divisional point on the sine *S. R.*; to wit the point *b.* on the arc, through the point *b.* on the sine; the point *c.* on the arc, through the point *c.* on the sine, and so on, in like manner; each successive point in the arc through each successive point in the sine. But, further, the point *F.* at the extremity of the octant having arrived at *S.* the extremity of the sine, let the quadrant be rolled until the same point *F.* become in contact on the line *B. E.*, then, since the motion of the arc is uniform, and the motions of all the points on the arc, in descending from the line *M. S.* to the line *B. E.*, and ascending from the line *B. E.* to the line *M.S.*, are similar and equal, the place of contact must necessarily be the point *Y.*, at the same distance from the point *T.* as the distance of the point *a.* from the point *M.* and the same as the distance of the point *b.* from the point *a.* And if the quadrant be again rolled back until the extremity thereof be in contact at the extremity *B.* of the line *B. E.*, the point *F.*, leaving the place of contact *Y.* on the line *B. E.*, must evidently pass through the point *S.* at the extremity of the sine; and the next divisional point *i.* on the arc, leaving the place of contact *T.*, must pass through the sine *S. R.* at the point of intersection *i.*; and, in like manner, each successive divisional point of the arc must pass through the corresponding successive divisional point in the sine; finally the point *a.*, leaving the point of contact must arrive at the extremity *R.* of the sine, andthe point *M.*, leaving *O.* on the line *B. E.*, must return to its first position, at the distance *a. M.* from the extremity *R.* of the sine. Since, therefore, the octant *M. F.* is divided into ten equal parts, of which equal parts, the part contained between the points *M.* and *a.* is one part; and since, if the quadrant be rolled, the point *a.* in the octant intercepts one extremity of the sine *S. R.*, and, if the rolling be continued the point *F.*, at the extremity of the octant, intercepts the other extremity of the sine, it is made manifest that the sine *S. R.* contains nine equal parts, each of them equal to each of the divisional parts of the octant *M. F.*, and that, since the remaining equal part *M. a.* of the octant is also shown to equal *M. a.* on the line *M. S.*, the line *M. S.* contains ten equal parts, each of them equal to each of the ten equal parts contained in the octant *M. F.*

Wherefore it is demonstrated that, if an octant be divided into ten equal parts, the sine of the octant shall contain nine equal parts, each of them equal to each of the equal parts contained in the octant. Q.E.D.

Corollary.—Hence the difference between the arc-length and the sine of the octant is equal to one-tenth of the octant.

THE FUNDAMENTAL THEOREM—Repeated (1).

The difference between the arc-length and sine of an octant is one-tenth the arc-length.

DEMONSTRATION by the Fundamental Construction—(Figs. 10 & 11, *Frontispiece.*)

Fig 10 is a repetition of Fig. 1 on an enlarged scale, similar letters denoting similar parts.

Divide the radius A.M. into ten equal parts. From the extremity M. take one part *A.*M. as a radius, and with centre *A.* and radius *A.B.* (equal to *A.*M.,) describe the quadrant *B.M.F.* Complete the lesser figure similarly to the greater figure.

Since *M.F.* the lesser and M.F. the greater are both octants, and the lesser radius *A.B.* is one-tenth the magnitude of A.B. the greater, the lesser octant *B.F.* is equal to M.*a.* the one-tenth part of the greater octant M.F.*;* therefore, if the lesser quadrant be rolled upon the line *B. Q.* until it become in contact thereon, the extremity *F.* of the lesser octant *M.F.* must necessarily arrive at *a.* the first point of intersection on the line M.S. the greater. Now *M.S.* the lesser, which is by the construction one-tenth the magnitude of M.S. the greater, contains *R.M.* the lesser (which, by the construction, is likewise one-tenth the magnitude of R.M. the greater), and contains also *R.S.* the lesser (which is one-tenth the magnitude of R S. the greater). Let the magnitude of *R.M.* the lesser, be doubled, and together with it the other parts of the figure. Then *R.S.* the lesser will be also doubled, and since *M.R.* and *R.S.* compound *M.S.*, *M.S.* the lesser is doubled by doubling *R.M.* the lesser. Now, by the construction, *M.S.* the lesser is one-tenth the magnitude of M.S. the greater, and therefore *M.S.* the lesser, doubled, is one-fifth the magnitude of M.S. the greater. Similarly, and by a parity of resaoning, if *R.M.* the lesser be increased to five times the magnitude, *M.S* will be increased five times, and become equal to the one-half of M.S. the greater. Therefore, if *R.M.* the lesser be increased ten times, *M.S.* the lesser will become M.S. the greater. But as *R.S.* the lesser is one-tenth of R.S. the greater, *R.S.* increased nine times will equal nine-tenths of R.S. the greater, and *M.S.* the lesser increased nine times will equal nine-tenths of M.S. the greater. Therefore nine-tenths of *R.S.*, and nine-tenths of *R.M.* are together equal to nine-tenths of M.S. Wherefore it is made manifest that R.M. is the one-tenth part of M.S.

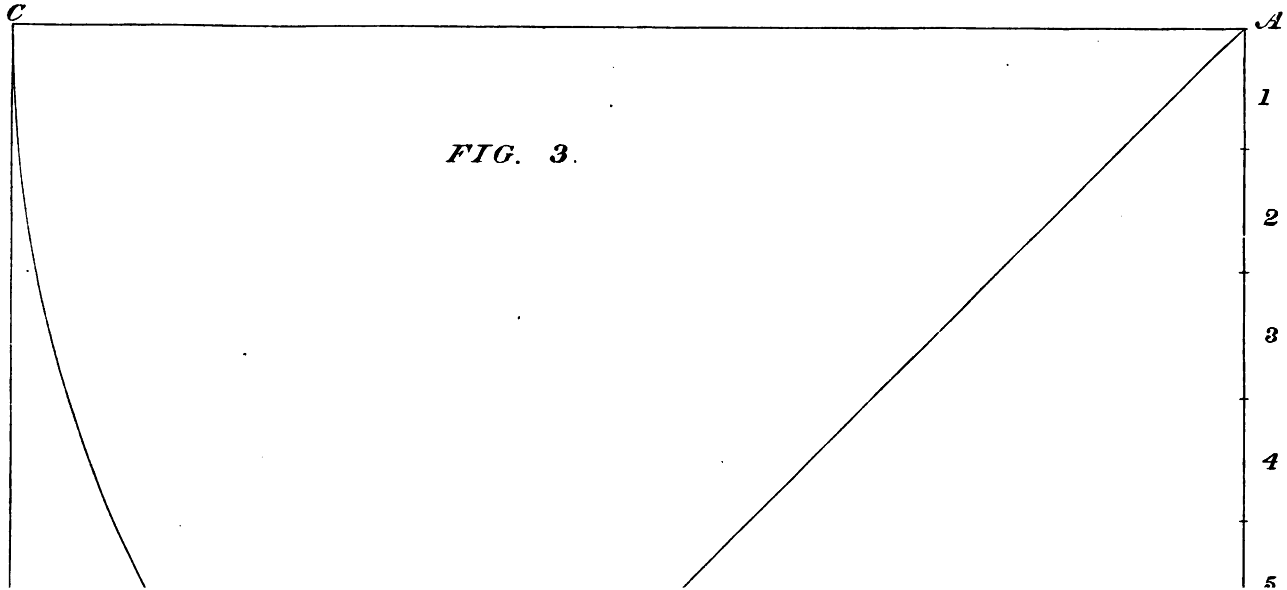
The Fundamental Theorem—Repeated (2.)

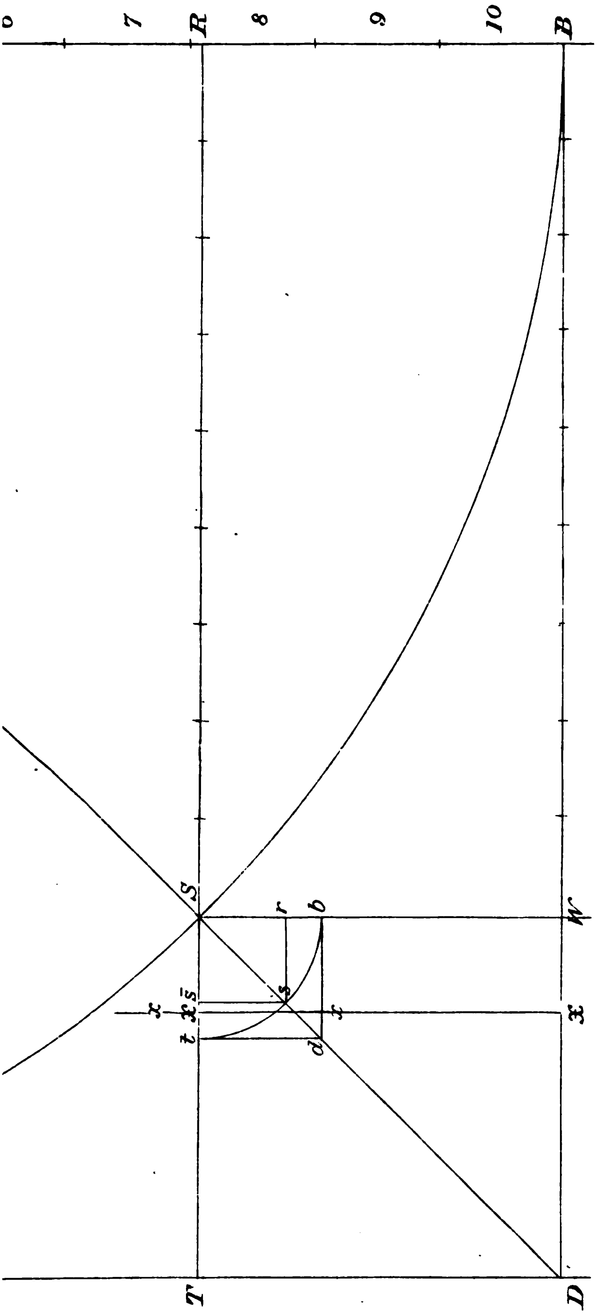
If the sine of an arc containing one-eighth of a circle be divided into nine equal parts, the difference of the sine and arc-length is equal to one of those nine equal parts.

Fig. 3. The Construction.—With centre *A.* and radius *A. B.* describe the quadrant *B. C.* Bisect *B. C.* in the point *S.* Draw the tangent *B. D.*, and the cotangent *C. D.*, of the octant *B. S.* Draw the sine *R. S.* of the arc *B. S.*, and produce *R. S.* through *S.* intercepting *C. D.* in the point *T.* Divide the sine *R. S.* into nine equal parts, and on the line *S. T.* take *S.X.* equal to one of the nine divisional parts of *R. S.* Draw the perpendicular *X.X.* intercepting the line *B. D.* in the point *X.* Draw also *S. W.* parallel to *X.X.*, intercepting the line *B. D.* in the point *W.*

Divide the radius *A. B.* into ten equal parts. From the point *S.*, parallel to the radius *A.B.*, draw *S. b.* equal to one-tenth of the radius *A. B.* With centre *S.* and radius *S. b.* describe the quadrant *b. t.* and bisect *b. t.*

FIG. 3.

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in the point *s*. Draw the tangent *b. d.*, the cotangent *t. d.*, and the sine *r. s.* From s. draw *s. z.* intercepting the line *S. T.*, at right angles, in the point z.

DEMONSTRATION.—Because the lesser arc *b. s.* is one-tenth the magnitude of the greater similar arc *B. S.*, the sine *s. r.* of the lesser is one tenth the magnitude of the sine *S. R.* of the greater; and *z. x.* the diff. of the sine and arc-length of the lesser is one tenth the magnitude of *S. X.* the diff. of the sine and arc-length of the greater; and *z. t.* the diff. of the sine and tangent of the lesser is one-tenth the magnitude of *S. T.* the diff. of the sine and tangent of the greater; and *x. t.* the diff. of the arc-length and tangent of the lesser is one-tenth the magnitude of *X. T.* the diff. of the arc-length and tangent of the greater; therefore *S. x.* the arc-length of the lesser is likewise one-tenth the magnitude of *R. X.* the arc-length of the greater. But *S. x.* the arc-length of the lesser arc *b. S.*, is also the diff. of the sine and arc-length of the greater arc *B. S.*; and therefore if the arc-length *R. X.* of the greater arc, be divided into ten equal parts, the sine *R. S.* contains nine of those equal parts, and *S. x.*, the diff. of the sine and arc-length, contains one of those equal parts. *Wherefore it has been demonstrated that* . . if the sine of an arc containing one-eighth of a circle be divided into nine equal parts, the difference of the sine and arc-length is equal to one of those equal divisional parts. *Q. E. D.*

The same demonstration shown negatively.

Let it be supposed possible that *S. X.* the difference of the sine and arc-length of the octant *B.S.*, is less than the one-tenth of *R. X.* the arc-length. Now it is manifest that *z. X.* the diff. of the lesser sine and arc is one-tenth of *S.X.* the diff. of the greater sine and arc, because by the construction the lesser arc *b.s.* is one-tenth the magnitude of the greater arc *B.S.* But, since the *diff.* of the arc and sine, and the *sine* itself, have always the same proportion each to the other and each to the arc, whatever the magnitude of the arc may be, a reduction in the magnitude of the *diff.* to the one-tenth is also a reduction in the magnitude of the *sine* to the one-tenth. Now by the supposition, *S.X.* in the lesser figure, containing the *sine* and the *diff.* of the reduced magnitude is (proportionally) greater than *S.X.* in the greater figure, which contains the *diff. only* of the greater magnitude.* Let the magnitude of *S.X.* in the lesser figure be again increased to ten times, evidently the whole of the lesser sine, which is included therein, will contain and become included in *S.R.* the greater sine, which is ten times the magnitude of the lesser sine; and *X. z.*, of the lesser figure, increased to ten times the magnitude, cannot become greater than *X.S.* of which it is the one-tenth. Therefore, it would follow that . . although *S. z.* is the one-tenth of *R. S.*, and *z. X.* is the one-tenth of *S. X.* . . *S. z.* and *z. X.* added together (*i. e. S. X.*) are less than equal to one-tenth of *R. S.* and *S. X.* added together, to suppose which would be absurd; therefore it cannot be held that *S. X.*, the diff. of the sine and arc-length, is less than one-tenth of *R. X.* the arc-length.

Again, let it be supposed possible that *S. X.* may be a magnitude greater than one-tenth of *R. X.*, then it would follow that . . although *S. z.* is manifestly the one-tenth of *R. S.*, and *z. X.* the one-tenth of *S. X.* *S. z.* and *z. X.* taken together (*i. e.*, *S. X.*) are greater than one-tenth of *S. R.* and *S. X.* taken together (*i. e.*, *R. X.*), to suppose which would be unreasonable; therefore since *S. X.* the diff. of the sine and arc-length, cannot be a lesser or a greater magnitude than the one-tenth part of *R. X.*, it must be equal to the one-tenth of *R. X.* the arc-length of the octant *B. S.*

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* Because *S.X.* in the lesser figure is one-tenth of *R.X.* in the greater, and, by the supposition, the *diff. only* is less than one-tenth.

THE FUNDAMENTAL THEOREM—Repeated, (3.)

If the sine of an arc containing the eighth part of a circle be divided into nine equal parts, the diff. of the sine and arc-length is equal to one of those nine equal divisional parts.

Fig. 4 (*a*.)* THE CONSTRUCTION.—With centre *A.* and radius *A.B.* describe the quadrant *B.C.*; and bisect the quadrant in the point *M.* Draw the sine *M.N.*, the tangent *B.E.*, and the secant *A.E.* of the octant *B.M.* From the point *M.* draw the perpendicular *M.D.* intercepting the line *B.E.* in the point *D.* On the line *B.E.* take the point *O.* at a distance from *B.* greater than the distance of the point *D.* by one-ninth of the distance *B.D.*

Diminish the radius *A.B.* by the one-tenth part thereof and with the reduced radius *K.B.*, and the centre *K.* describe the lesser octant *B.n.* From *n.* draw the perpendicular *n.e.* intercepting the line *B.E.* in the point *e.*

THE REQUISITION.—Assume the distance of the point *O.* from *B.* to indicate the arc-length of the greater octant *B.M.* It is required to demonstrate the relation of the difference between the arc-length (indicated by *D.O.*,) and the sine of the octant *B.M.*, to the actual arc-length (or to the sine length) of that octant.

DEMONSTRATION.—Since the arc *B.n.* is of one-tenth less magnitude than the similar arc *B.M.*, the arc-length of *B.n.* is one-tenth less than the (actual) arc length of *B.M.* indicated by *B.O.* And since the sines of similar arcs of different magnitudes are to each other as the arcs to which they belong, the sine *n.p.* of the arc *B.n.* is one-tenth less then the sine *M.N.* of the arc *B.M.* The difference of the arc-lengths of the lesser and greater arcs, which, by the construction, is one-tenth of the arc-length of the greater arc, contains therefore one-tenth of the sine-length *B.D.* of the greater arc, together with one-tenth of the diff. between the sine-length and the arc-length of the same arc *B.M.* Therefore the actual arc-length of the arc *B.M.* contains ten-tenths of the sine-length together with ten-tenths of the diff. between the sine-length and arc-length. Now the diff. between the sine-length and the arc-length has been shown to contain one-tenth of the sine-length together with one-tenth of the diff. between the sine and arc-length. *Wherefore* the diff. between the sine-length and arc-length of the arc *B.M.* (indicated by *D.O.*), is one-tenth of the actual arc-length (indicated by *B.O.*), and consequently, if the sine-length *B.D.* be divided into nine equal parts, the diff. (*D.O.*), between the sine and arc length, is equal to one of those equal divisional parts. *Q. E. D.*

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* See page 37.

DEMONSTRATION TO THE LAST PROPOSITION—Repeated.

Now if the lesser arc *B. n.*, (or *B. m.*), be rolled upon the line *B. E.* until the point *n.*, at the terminal extremity of the arc, become in contact, the distance of the point of contact from *B.* must be one-tenth less than the similar point of contact *O.* of the extremity *M.*, of the greater arc; because the arc *B. n.* is of one-tenth less magnitude than the similar greater arc *B. M.*.. (4th axiom). But since the sines of similar arcs are to each other as the radii of the circles to which they respectively belong, the sine of the arc *B. n.* is one-tenth less than the sine of the arc *B. M.* Therefore one-tenth of *B. O.*, the arc-length of the greater arc, contains one-tenth of the sine *B. D.*, and one-tenth of *D. O.*, the difference of the sine and arc-length. Consequently *D. O.*, the difference of the sine and arc-length of the arc *B. M.*, is equal to one-tenth of the sine together with one-tenth of the difference of the sine and arc-length. Wherefore the difference of the sine and arc-length of an arc containing one-eighth of the circle, is shewn to be one-tenth of the entire arc-length.

Prop. A.—Theorem.

If an arc containing one eighth of a circle, be applied upon a straight line, and, from the terminal extremity of the arc, a perpendicular be drawn intercepting the straight line, and if from the arc one-tenth thereof be cut off, then if the remaining arc (to wit, the arc containing nine-tenths of the whole arc,) be rolled upon the straight line, until the terminal extremitg of the arc become in contact, the point of contact shall be the same point on the straight line intercepted by the perpendicular drawn from the terminal extremity of the whole arc.

Fig. 4 (*a*). Let the arc *B. M.*, containing one eighth of a circle and described with the radius *A. B.*, be the arc applied upon the straight line *B. E.*, and let *M. D.* be

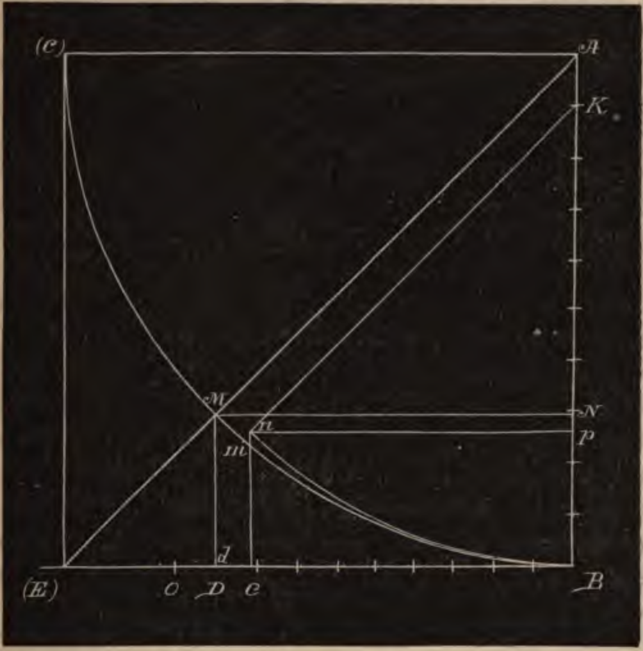

Fig. 4 (*a.*)

the perpendicular drawn from *M.* intercepting the straight line at *D.* And let the arc *B. m.*, nine-tenths the length of *B. M.*, be the arc cut off from *B. M.* If the arc *B. m.* be rolled upon the straight line until *m.* arrive at the straight line, the point of contact shall be the point *D.* intercepted by the perpendicular *M. D.*

From the radius *A. B.* take *K. B.* nine-tenths the length of *A. B.*, and with centre *K.* and radius *K. B.*, describe the arc *B. n.* similar to the arc *B. M.*, and equal in length to the arc *B. m.*[*] Draw *M. N.*, the sine of the arc *B. M.*, and *n. p.*, the sine of the arc *B. n.*, From *n.* at the extremity of the arc *B.n.* draw the perpendicular *n. C.* intercepting the line *B. E.* at *C.*

Let the arc *B. M.* be rolled upon the straight line until *M.* become in contact, and let *O.* indicate the point of contact. Let also the arc *B. n.* be rolled until *n.* become in contact on the straight line, and let *d.* indicate the point of contact.

Because the radius *K. B.* is nine-tenths of the radius *A. B.*, the sine *n. p.* is nine-tenths of the sine *M. N.*, and the arc-length of *B. n.*, which is indicated by *B. d.*, is nine-tenths the arc-length of *B M.*. which is indicated by *B. O.*, and *C d.*, the difference of the sine and arc-length of *B. n.*, is equal to nine-tenths of *D. O.* the difference of the sine and arc length of the arc *B. M.*

Now if the arc *B. M.* be rolled upon the straight line (back again) from the point of contact (indicated by *O.*) until the opposite extremity of the arc be in contact with the opposite extremity of the straight line at *B.* the point *M.* must again arrive at *M.* the extremity of the perpendicular *B. M.*, as in the first position of the arc, and, since the difference of the arc-length and sine of *B. n.*, has the same proportion to the difference of the arc length and sine of *B. M.*, which the arc *B. n.* has to the arc *B. M.*, therefore, if the arc *B. n.* be rolled upon the straight line (back again) from the point of contact indicated by *d*, until the opposite extremity of the arc be in contact with *B*, at the opposite extremity of the straight line *n.*, the point at the extremity of the perpendicular *n. C.*, at which the extremity *n.* of the arc arrives, must be arrived at by the point *n.* from a horizontal distance *C. d.* having the same proportion to *D. O.*, which the sine *n. p.* has to the sine *M. N.* Now the arc *B. n.* is equal to the arc *B. M.* diminished by one-tenth,* and *C. O.* is the difference(*C.d.*) of the sine and arc-length of *B. n.*, and the difference (*D. O.*) of the sine and arc-length of *B. M.*, taken together..that is, the difference of the sine of the lesser arc and the arc-length of the greater arc. But, because the ratio of *B. D.* to *B. C.* is the same as the ratio of *B. O.* to *B. d.*, and the same as the ratio of *d. O.* to *c. d.*, the part *C. D.*, of the whole distance *C. O.*, has necessarily the same proportion to the part *D. O.* of the whole distance which the sine *n. p.* has to the sine *M. N.*, and, therefore, the distance *C. d.* is the same as the distance *C. D.*, and the point indicated by *d.* is the same point *D.* at the extremity of the perpendicular *M. D.* *Wherefore it is demonstrated* that if an arc containing one-eighth of a circle, &c., &c. *Q. E. D.*

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[*] Circles are proportional to each other directly as their radii; therefore, since the radius *K. B.* is nine-tenths of *A. B.* the arc *B. n.* is nine-tenths the arc *B. M.* But the arc *B. m.* is nine-tenths of *B. M.*, therefore, the arc *B. n.* is equal to the arc *B. m.*

Corollary.—Hence it follows,—because the point *n.*, when the arc *B. n.* is rolled, becomes in contact on the straight line at the point intercepted by the perpendicular *M. D.*, and, because the distance *B. D.* is (consequently) nine-tenths of *B. O.*,—that the sine of an arc containing one-eighth of a circle is nine-tenths the arc length, † (*i.e.*, the ratio of the length of the sine to the arc-length is the ratio of nine to ten.)

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* That is, *B. M.* is the arc *B. n.* increased by one-ninth of *B. n.*

† Independent demonstration will be given of this proposition, but the corollary becomes apparent so soon as the point of contact, at which the extremity of the lesser arc arrives on the straight line, is determined.

Note.—Having regard to the great importance of the subject, and as this theorem is fundamental, it will be now repeated in a somewhat different form :—

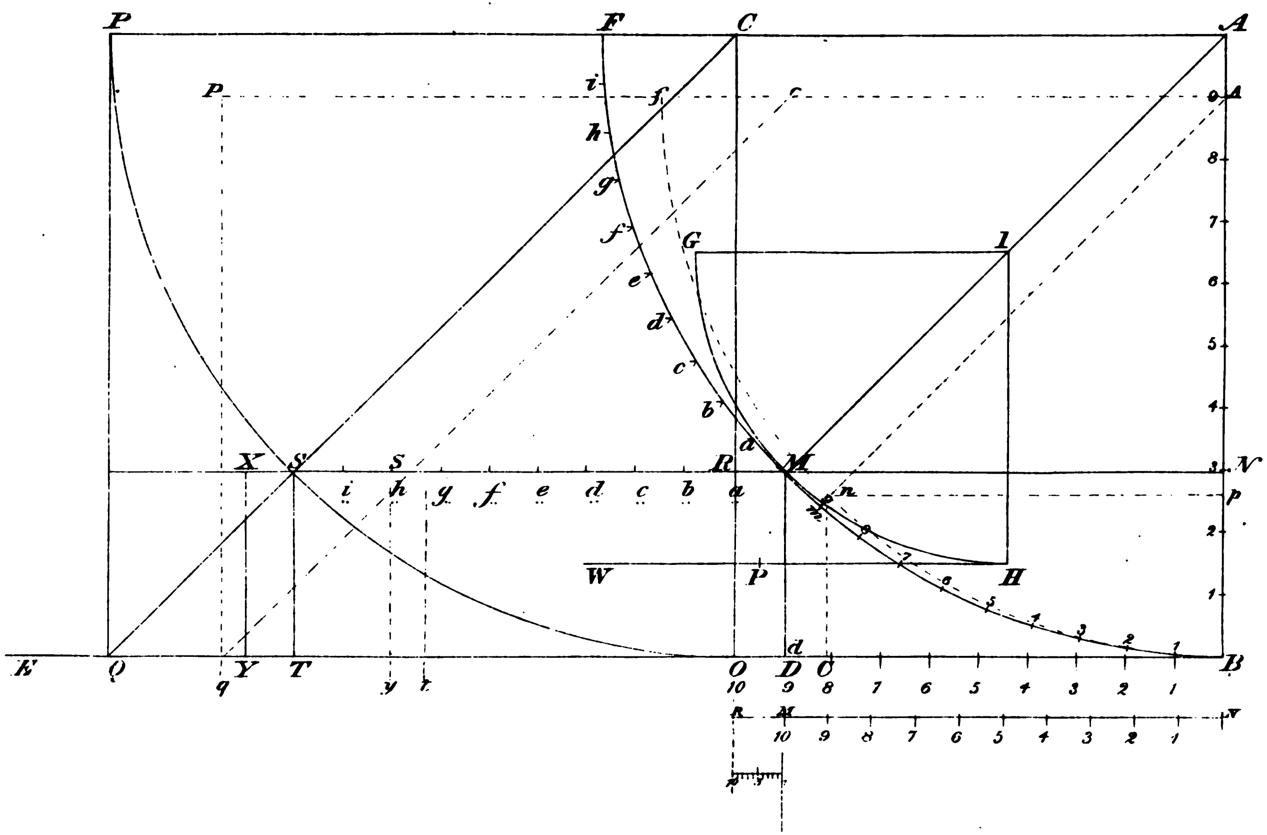
Scholium.—In considering the following case, since if several similar arcs of different magnitudes be rolled on a straight line, the motion of the terminal point at the extremity of each arc is compounded of the vertical and oblique motion of each and all the component parts of that arc, it is requisite only in comparing the results of the motions of two or more of the similar arcs with each other, to take the straight horizontal longitudinal advance—that is, the horizontal motion—into consideration; because the result of the horizontal advance includes the result of the vertical motion of which it is in part compounded, and which must be necessarily proportional to the horizontal advance and, also, to the magnitude of the arc.

Theorem A. (*repeated.*) Fig. 4 (*a*).

When the arc *B. M.* is rolled, *M.* advances to a point (*i. e.*, some point) on the straight line, indicated by *O.* When the arc *B. n.*, which is similar to the arc *B. M.*, and contains nine-tenths of the length contained by *B. M.*, is rolled upon the straight line, *n.* advances to a point (some point) indicated by *d.* Now the distances *O.* and *d.* from *B.* are so related to each other by the construction that *B. d.* contains nine-tenths the length contained by *B. O.* Also the sine *n. p.*, of the arc *B. n.*, is nine-tenths the sine *M. N.*, of the arc *B.M.*

(2.) The distance *D. O.*, which is the horizontal advance of the point *M.*, contains the difference between the sine and arc-length of *B. M.* Now, if *B. M.* be reduced by one-tenth part thereof, and the remaining arc, by diminishing the radius in the same proportion, is converted into a similar arc, containing one-tenth less length, the sine of the lesser arc *B. n.* thus described will have the same proportion to the sine of the greater

FIG. 4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arc *B. M.*,—to wit, the sine *n. p.* of the lesser arc will be nine-tenths the length of the sine *M. N.* The difference (*D. O.*) between the arc-length and sine of *B. M.* is to the difference (*C. d.*) between the arc-length and sine of *B. n.*, in the proportion of ten to nine. But the one-tenth by which *D. O.* exceeds *C. d.*, is manifestly the increase belonging to the one-tenth of the arc *B. M.* by which that arc exceeds the arc *B. n.* Therefore, when the arc *B. M.* is diminished by one-tenth thereof, if the lesser arc, thence resulting, is rolled until the terminal point become in contact on the straight line, the advance

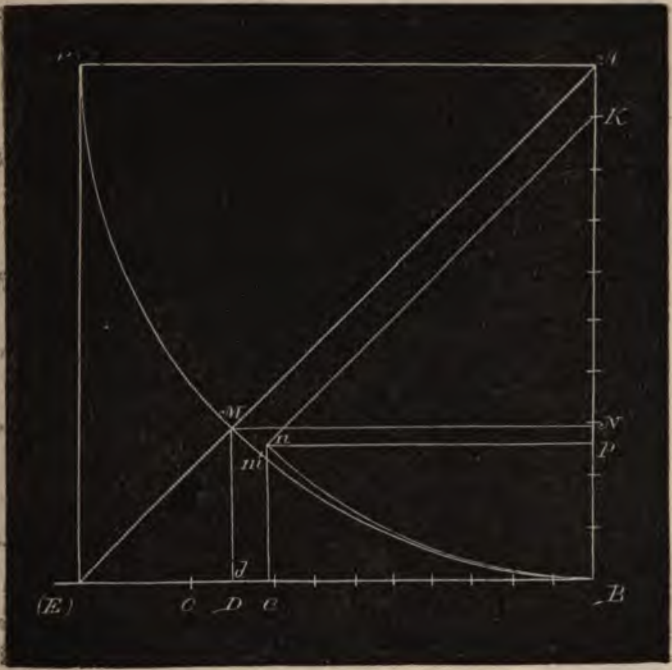


Fig. 4 (*a*).

of the terminal point must be less than *D. O.* by one-tenth of *D. O.* But the sine of the arc *B. n.* is one-tenth less than the sine of the similar arc *B. M.*, therefore the distance between *C.* and *d.* must be less than the distance *C. O.* by . . one-tenth of *D. O.* and one-tenth of *B. D.* taken together; that is by the distance *D. O.*; because *B. D.* and *D. O.* taken together equal *B. O.*, and *B.d.* is nine-tenths of *B. O.*; therefore *B. d.* is equal to *B. O.* diminished by *D. O.*; but the distance *B. O.*, diminished by the distance *D. O.*, is *B. D.* and therefore the distance *B.d.*, is the same as the distance *B.D.* Wherefore it is demonstrated that the point of contact indicated by *d.* on the straight line *B.E.*, at which the terminal point of the arc *B.n.* arrives, is the point intercepted by the perpendicular *M.D.* drawn from the terminal point *M.* of the greater arc *B.M.*

(3) Because:—

$$\left.\begin{array}{r} B.C. : B.D. :: B.d. : B.O. \\ \text{and } B.C : B.d. :: B.d. : B.O. \end{array}\right\} \begin{array}{l} \text{therefore} \\ B.\,d. = B.\,D. \end{array}$$

That is:—

The sine of *B.n.* is to the sine of *B. M.* as the arc-length of *B.n.* is to the arc-length of *B. M.*

The sine of *B.n.* is to the arc-length of *B.n.* as the arc-length of *B.n.* is to the arc-length of *B.n* increased by one-ninth.

Wherefore the arc-length of *B.n.* equals the sine of *B.M.*

It immediately follows—because *d.O.* is one-tenth of *B.O.*, that *D.O* (which is the same) is also the one-tenth of *B.O.*

(4.) Again; let it be assumed to be possible that *n.* advances to some point, *d.*, less distant than *D.* from *B.*, then must the ratio of the distance *B.O.*, to the distance *B.d.*, be greater than the ratio *M.N.*, to *n.p.*, because the point *D.* is, by the assumption, in advance of the point *d.*, and the greater arc must advance from (its starting point) the point *D.* proportionally to the advance of the lesser arc from (its starting point) the point *C.* Now the distance *B.O.* contains the distance *B.C.*, together with the advance of the two arcs taken together—to wit, the advance of *M.* from *D.*, together with the advance of *n.* from *C.*, and together with the difference, if there be any, between *d.*, the advanced place to which *n.* advances, and *D.* the place from which *M.* commences to advance—that is, if *d.* be less distant than *D.* from *C.*, *C. O.* includes the diff. *D. O.* together with the diff. *C.d.* and together with the diff. *d.D.*; (thus ∴ *C.O.* includes the advance of *n.* through *C. d.*, of *M.* through $C.d. + \frac{C.d.}{9}$, and the distance *d.D.*) But *B. C.* and *C.d.* taken together equal *B.d.*, which is equal to *M.N.* And *M.N.* is to *n.p.* as ten to nine. Therefore $B.d. + C.d. + \frac{C.d.}{9}$, is to $B.C. + C.d.$, as ten to nine.

By the assumption *B.O.* (*i.e.*, $B.C + C.O.$) contains $B. d. + \frac{B.d.}{9} + d.D.$, and therefore the ratio of *B.O.* to *B.d.* is greater than the ratio of *M.N.* to *n.p.* by the distance *d.D.* Now, it is manifestly impossible that *B.M.* can increase by advancing in a greater ratio of proportion to the increase of *B.n.* than the proportion of the sine *M.N.* to the sine *n.p.*, because *B.M.* and *B.n.* are similar arcs; therefore the distance *B.d.* cannot be less than the distance *B.D.* By similar reasoning it may be shown that the distance *C.d.* cannot be greater than *C.D.*, because then the advance of the arc *B.M.* would be proportional to the advance of the arc *B.n.* in a ratio of proportion less than the ratio of the sine *M.N.*, to the sine *n.p.*, to suppose which would be absurd. Wherefore it is demonstrated that the point of contact of the lesser arc *B.n.* on the straight line *B.E.*, indicated assumptively by *d.*, is the same point *D.* at the extremity of the perpendicular *M.D.* *Q.E.D.*

(5) Again —Fig. *b* (repeats the construction of Fig. *a*.) Produce *D. M.* through *M.*, and make *F. D.* equal to *A.B.* With centre *F.* and radius *F.D.* describe the arc *D.N.* equal and similar to the arc *B.M.*, produce the straight line *E.B.* through *B.* indefinitely, and upon the line so produced roll the arc *D. N.* from *D.* in the direction *D.B.* until the extremity *N.* of the arc become in contact upon the line at some point *T.* Now the distance *B.T.* is manifestly equal to the distance *D.O.*, because if the arc be rolled upon the line to the extremity of the arc opposite to the point of contact, the advance in that direction must necessarily be equal to

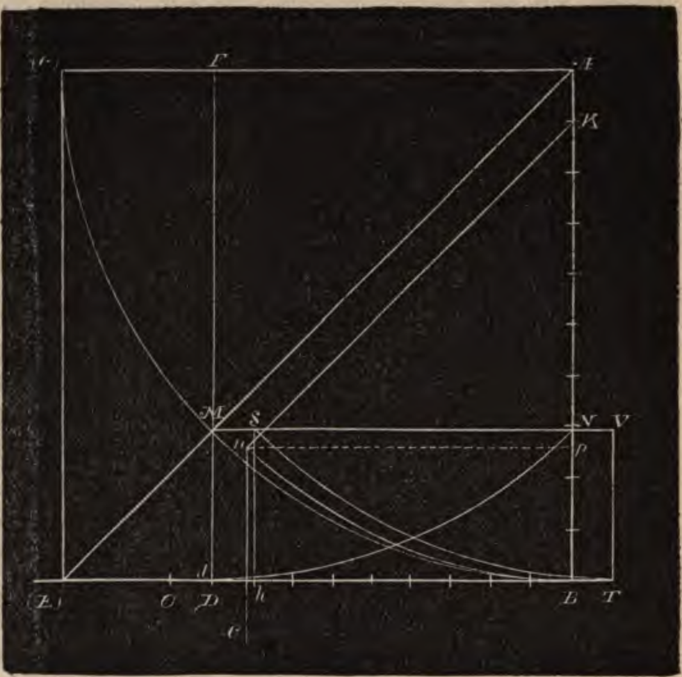


Fig. 4 *(b.)*

the advance in the opposite direction, if the arc be similarly rolled in the opposite direction; but the other extremity of the arc—to wit, the extremity which is in contact with the line at *D.*, must likewise advance (to *S.*) in the direction *D.B.*, a horizontal distance *D.h.* equal to the distance *B.T.;* manifestly the arc *T. S.* must be equal to the arc *D. N.* (because both are the same arc in an altered position.) Also *D. h.*, which is equal to *D.O.*, has the same ratio to *C.d.* which *M.N.* has to *n.p.*—to wit, the ratio of ten to nine. And the arc *S. T.* has likewise the same ratio to the arc *B.n.*, and the line *D.T.* the same ratio to the line *B.d.*

Therefore (*d.B.* + *B. T.*) = *d. T.* = (*B. d.* + *d.O.*) = (*B.O.*) Now *B.O.* contains *B.C.* + *C.D.* + *D.O.*, and *d.T.* contains *B.T.* + *B. C.* + *C.d.*, and since *B. T.* equals *D.O.*, *C.d.* must equal *C.D.* Wherefore it is demonstrated, &c., &c.

Note.—This last method is given as furnishing an independent means of testing and verifying the preceding demonstrations.

PROP. B.—Theorem.

That if a straight line be drawn from the circumference of a circle to the radius, at right angles to the radius, such that one-eighth part of the circle be contained between the point in the circumference intercepted by the line and the point intercepted by that extremity of the radius nearest to it, then if the one-eighth part of the circle be divided into ten equal parts, the straight line so drawn shall contain nine equal parts, each of them equal to each of the ten equal parts contained in the one-eighth part of the circle.

Fig.4. (*c.*)—Let *M.N.* be the straight line intercepting the radius *A. B.*, at right angles, in the point *N.*, and bisecting the quadrant *B. F.*, in the point *M.* The straight line *M. N.* shall contain nine equal parts, each of them equal to each of the ten equal parts contained in the octant *B.M.*

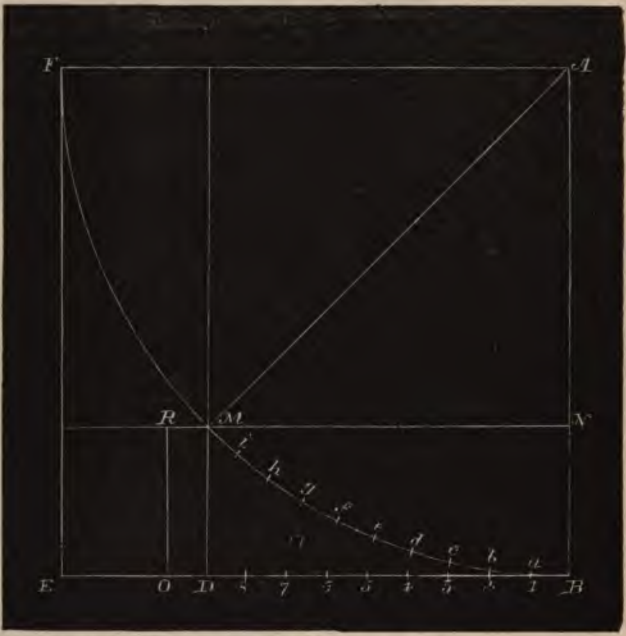


Fig. 4. (*c.*)

From the point *B.* draw the tangent *B. E.*; produce the line *N. M.* through *M.* indefinitely; and from the line *B. E.* cut off *B. O.* greater than *B. D.* by the one ninth of *B. D.*; divide the arc *B. M.* into ten equal parts, at the points of equal division *a. b. c. d. e. f. g. h. i.*

Because the line *B. O.* is equal to the arc *B. M.* (*Theorem A.*,) and the line *B. D.* contains nine-tenths of *B. O.*, therefore the line *N. M.*, which is manifestly equal to *B. D.*, also contains nine-tenths of the arc *B. M.* *Wherefore it is demonstrated* that if a straight line be drawn, &c., &c., *Q. E.* .

Corollary.—*Hence*—since if the line, drawn from the

circle, intercepting the radius on the one side, be drawn from the circle at right angles through the radius until it intercepts the circle on the other side of the radius, the part of the line on the one side of the radius is necessarily equal to the part of the line on the other side, and the fraction of the circle, contained between the point of interception of the line on the one side and the extremity of the radius, is equal to the similar fraction of the circle cut off between the extremity of the radius and the point of interception of the line on the other side—*it follows that,* if a straight line be drawn from a circle through the radius, at right angles to the radius, until the line intercept the circle at the opposite side of the radius, and the line be so drawn that the part of the circle intercepted between the two extremities of the line be the one-fourth part of the circle, then—if the straight line so drawn be divided into nine equal parts, the fourth part of the circle shall contain ten equal parts, each of them equal to each of the nine equal parts contained by the line, and the whole circle shall contain forty such equal parts.

PROP. C.—PROBLEM.

Requisition.—It is required upon a given straight line to describe an arc, such that the arc shall be equal in length to the given straight line, and shall contain a definite fraction of a circle of definite magnitude.

Definition.—Fig. 5. Let *B. O.* be the given straight line : it is required upon *B. O.*, to describe a definite arc, equal in length to *B. O.*

Construction.—Divide the line *B. O.* into ten equal parts at the points of equal division *a. b. c. d. e. f. g. h. i.* From the point *B.*, at the extremity of the straight line, draw the perpendicular of indefinite length *B. ë.*, and from the point *i.*, on the straight line *B. O.* draw the perpendicular of indefinite length *i. H.* On the line *B. ë.* take *B. d'.*, equal to *B. i.* and join *i. d'.* From *B.* at right angles to *i. d'.*, draw the line of indefinite length *B.E.*, and on the line *B. E.* take *B. C.* equal to *B. i.* Produce *B. O.* indefinitely through *O.* and, through the point *C.*, draw a line at right angles to *B. E.*, intercepting *B. ë*, at *A.* and intercepting the production of *B. O.* at *D.* With centre *A.* and radius *A. B.* describe the quadrant *B. M. F.* bisected at *M.* by the perpendicular *i. H.* From the point *M.*, and at right angles to *A. B.*, draw *M. N.*, intercepting *A. B.* at *N.*

Result.—The arc *B. M.* described with the definite radius *A. B.* shall be the required arc, equal in length to the given straight line *B. O.*

Demonstration.—Because the arc *B. M. F.* is a quadrant, and the point *M.* bisects the quadrant, therefore the arc *B. M.* contains the eighth part of a circle.

Now *M. N.* is the sine of the arc *B. M.*, and *M. N.* is manifestly equal to *B. i.* on the line *B. O.*; but *B. i.* contains nine of the ten equal divisional parts of the line *B. O.*, and it has been demonstrated (Prop. A.) that, if the sine, of an arc containing the eighth part of a circle, be divided into nine equal parts, the arc contains ten equal parts, each of them equal to each of the nine equal parts contained by the sine. Therefore *B. M.* is equal to *B. O.*

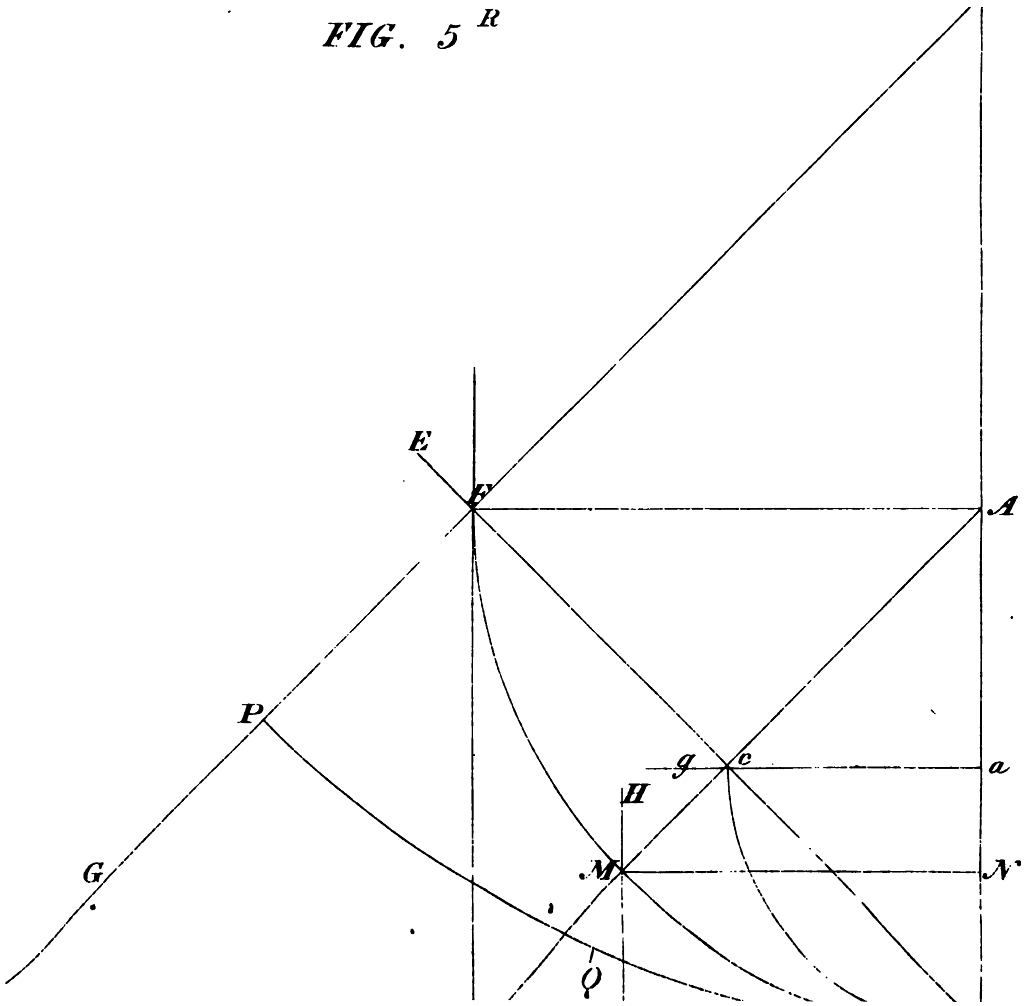
Wherefore the arc *B. M.* containing the eighth part of a circle has been described with the definite radius *A. B.* upon the given straight line *B. O.*, and has been shown to be equal in length to the given straight line, as was required to be done.

Supplement 1.—*To describe upon the given straight line a quadrant equal in length to the given straight line.*

(Fig. 5$^{R.}$) Bisect the radius *A. B.* at the point *a.*, and from *a.* at right angles to *A. B.*, draw the line of indefinite length *a. g.* With centre *a.* and radius *a. B.* describe the quadrant *B.c.*, intercepting *a.g.* in the point *c.*

Now because *a. B.* is the one half of *A. B.*, the quad-

FIG. 5 R



rant *B. c.* is the fourth part of a circle of half the magnitude of the circle of which the arc *B. M.* (Fig. 5.) is the one eighth part; consequently the quadrant *B. c.* is equal in length to the arc *B. M.*; and it has been demonstrated that the arc *B. M.* is equal in length to the line *B. O.*, which is the given straight line. Wherefore the quadrant *B. c.* is also equal to the given straight line. (*See Note.*)

Supplement 2.—*To describe upon the given straight line an arc containing the sixteenth part of a circle such that the arc shall be equal in length to the given straight line.* (*See Note.*)

(Fig. $5^{R.}$) Produce *A. B.* through *A.*, and from the production take *R. B.* twice the length of *A. B.* From *R.* through *F.* draw *R. F. G.* of indefinite length. With centre *R.* and radius *R. B.* describe the arc *B. P.*, intercepting *R. F. G.* at *P.* Bisect the arc *B. P.* at *Q.* Now, since the arc *B. Q.* is the one half of the arc *B. P.*, which contains the eighth part of the circle;* and, because the radius *R. B.* is twice the length of the radius *A. B.*; the arc *B. Q.* contains the sixteenth part..of a circle twice the magnitude of the circle of which the arc *B. M.* contains the eighth part. Therefore the arc *B. Q.* is equal in length to the arc *B. M.*, and it has been shown that the arc *B. M.* is equal in length to the given straight line *B. O.* Wherefore the arc *B. Q.*, which contains the sixteenth part of a circle, is also equal to the given straight line.

Note.—The constructions and demonstrations to these two supplementary problems are based upon the theorem or supposed axiom (at the present time accepted and authorized by mathematicians)...that, if the arc of a circle be bisected and the resulting half-arc be radially duplicated in magnitude, † the quantity of length contained in the duplicated half-arc is equal to the quantity of length contained in the original arc. Now this theorem we refuse to accept; and we assert that our demonstration of the relation of the circle to the straight line herein contained, includes demonstration that the theorem is unsound and untenable. We therefore expressly caution the reader that the solutions to these two supplementary problems, as given above, are not, in fact, sound; but we have, under the peculiar circumstances, allowed them to stand, with this caution, as instructive illustrations of the general relationship.

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* Because the secant *R. F. G.* is parallel to the secant *A. M. D.*, the arc *B. P.* is similar to the arc *B. M.*, which is half the quadrant *B. F.*, and therefore *B. P.* is half a quadrant.

† That is...increased in magnitude by duplication of the radius, or to define the expression more precisely, if there be substituted for the half-arc

Explanation of the nature and of the quantity of longitudinal difference between the radially duplicated half-arc and the original arc from which it was cut off, will be found in Parts Second and Third, under the titles...Process of continual bisection and radial duplication; and..The Theory of curvature.

PROP. D.—PROBLEM.

Requisition.—Through a given point in the circumference of a given circle it is required to draw a straight line tangential to the circle, which shall be equal in length to the given circle.

Definition.—Fig. 6.—Let *B. E. G. H.* be the given circle, and let *B.* be the given point,—it is required to draw a straight line through the point *B.* which shall be equal in length to the circle *B. E. G. H.*

Construction.—Through the point *B.* draw the line *d. B. a.* of indefinite length, and from *B.* perpendicular to *d. B. a.* draw the diameter *B. G.*; bisect *B. G.* at *C.*, the centre of the circle; and, through *C.* at right angles to *B. G.*, draw the diameter *E. H.* Bisect the quadrant *B. E.* at *M.*; and, at right angles to *B. C.*, draw the line *M. O.*, intersecting *B. C.* at *N.*, and intercepting the circle and bisecting the quadrant *B. H.* at *O.*

Divide the chord *M. O.* of the quadrant *M. B. O.* into eighteen equal parts. On the line *d. B. a.* take, from *B.*, in the direction *B. d.*, *B. f.* containing ten equal parts

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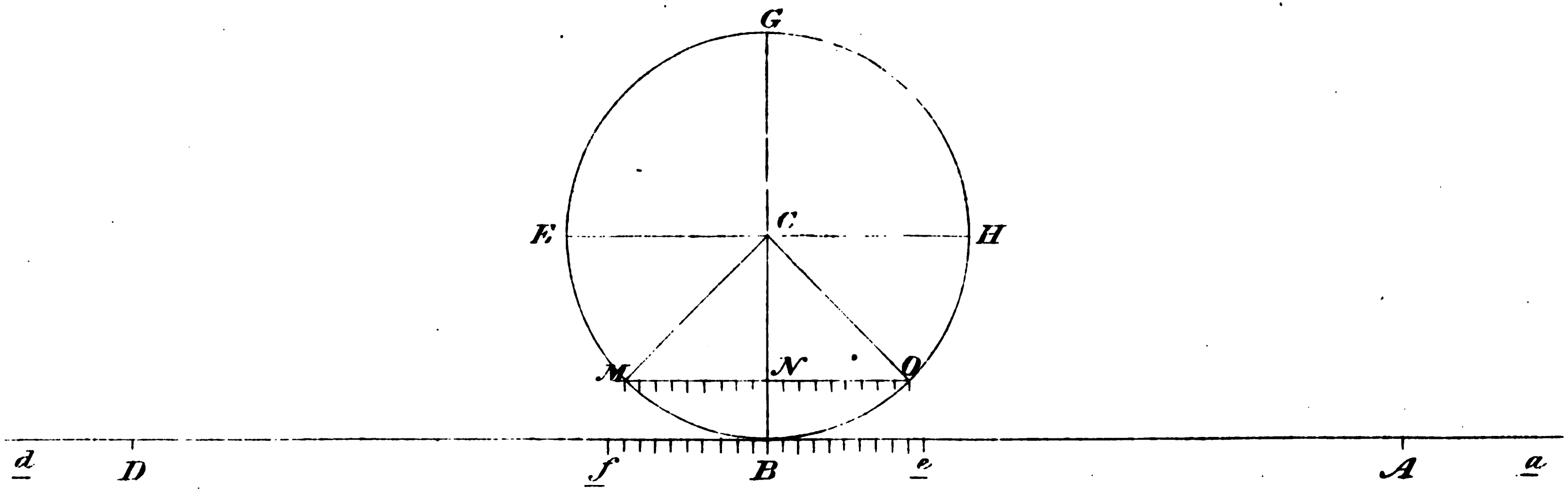
a similar fraction (half-arc) of a circle double the magnitude of the circle to which the original arc belongs.

FIG. 6.



each of them equal to each of the eighteen equal parts contained in the chord *M. O.* (Theor. B.,) and on the same line *d. B. a.*, from *B.* in the opposite direction, take *B. c.* containing likewise ten equal parts, each of them equal to each of the eighteen equal parts contained in the chord *M. O.* Increase *B. f.* through *f.* on the line *B. d.*, and take *B. D.* equal to four times the length of *B. f.*, and on the same line in the opposite direction increase *B. e.* through *e.*, and take *B. A.* equal to *B. D.*

Result.—The straight line *D. B. A.* contained between the points *D.* and *A.* shall be the required straight line equal in length to the circle *B. E. G. H.*

Demonstration.—Because the point *M.* bisects the quadrant *B. M. E.*, and because the sine *M. N.*, of the arc *B.M.*, is divided into nine equal parts, therefore *B.f.*, part of the line *d. B. a.*, containing ten equal parts each of them equal to each of the nine equal parts contained in the sine *M. N.*, is equal in length to the arc *B. M.* (Prop. B.) But the part *B. e.* of the same line *d. B. a.* is equal to the part *B.f.*, and is similarly related to the arc *B.O.*, which is similar and equal to the arc *B.M.*; therefore *f. B.e.*, the two parts of the line taken together, is equal to the quadrant *M. B. O.* containing the two arcs taken together. Now the line *D. B. A.* is (by the construction) equal to the part thereof *f. B. e.*, taken four times together, and the circle *B. E. G. H.* is equal to the quadrant *M. B. O.* taken four times together; therefore the line *D. B. A.* is equal to the circle *B. E. G. H.*

Wherefore the line *D. B. A.* drawn through the given point *B.* in the circumference of the circle, and tangential to the circle, is equal in length to the circle *B. E. G. H.*

Q. E. D.

Corollary.—Hence it follows that, if the chord of a quadrant be divided into (9) eighteen equal parts, a straight line containing (10) twenty equal parts, each of them equal to each of the (9) eighteen equal parts contained in the chord, is equal in length to the quadrant.

Wherefore, if a square be inscribed in a circle, the ratio of the inscribed square to the circle is the ratio of nine to ten.

PROP. E.—PROBLEM.

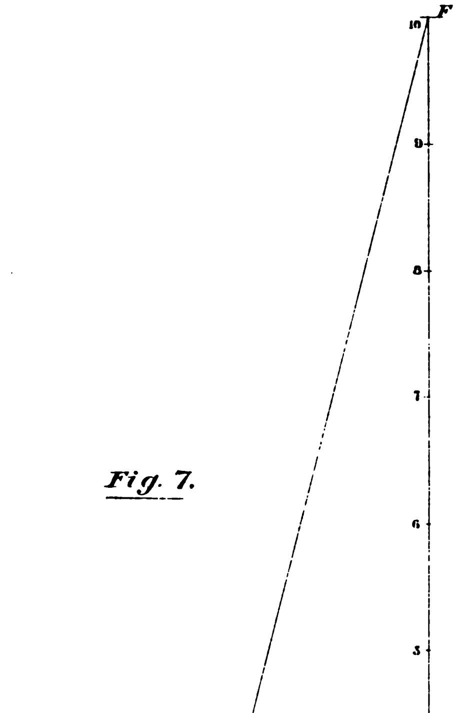
It is required to cut off from a given straight line a part thereof equal to the given definite arc of a circle.

Fig. 7. Let the octant *B.M.* described from the centre *A.* with the radius *A.B.* be the given definite arc, and let *B.E.* of indefinite length be the given straight line. It is required to cut off from the line *B.E.* a part thereof equal in length to the octant *B.M.*

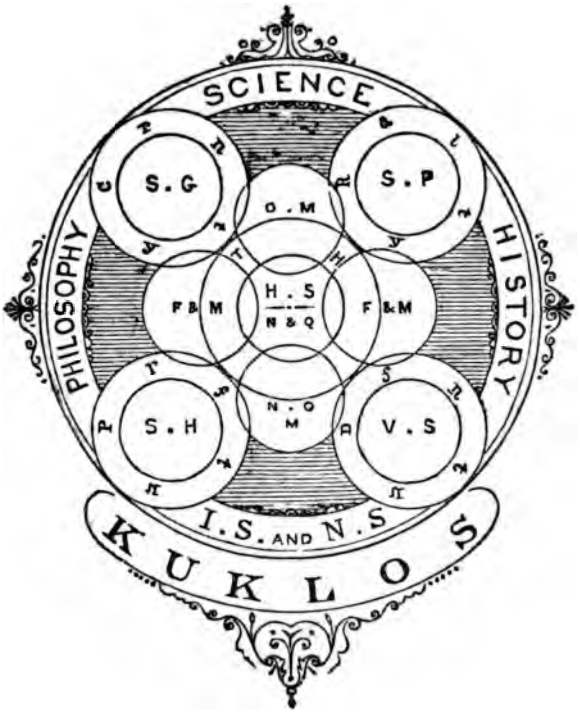
From *M.* draw the perpendicular *M.D.* intercepting *B.E.* at the point *D.* Draw also the sine *M.R.* of the octant *B.M.* Produce the radius *B.A.* through *A.*, and make *B.F.*, the production of *B.A.*, ten times the length of the versed sine *B.R.* From *F.*, through *M.*, draw the line *F.M.O.*, intercepting the line *B.E.* in the point *O.* *B.O.* shall be the required line equal in length to the octant *B.M.*

Because the angles *F.R.M.* and *F.B.O.* are right angles, and the triangles *F.M.R.* and *F.O.B.* to which they belong are similar triangles, the base *B.O.* of the one has the same ratio to the base *R.M.* of the other, which the greater line *F.B.* has to the lesser line *F.R.* Now the ratio of *F.B.* to *F.R.* is by the construction, that of ten to nine; therefore, the ratio of the line *B.O.* to the line *R.M.* is that of ten to nine. But the line *R.M.* is the sine of the octant *B.M.*, and if the sine of the octant be divided into nine equal parts, the arc-length of the octant contains ten equal parts, each

Fig. 7.



equal to each equal part of the sine. (Fundamental Theorem). Wherefore the line *B.O.* has been cut off from the line *B.E.*, and is shown to be equal in length to the arc-length of the given definite arc *B.M.*, as was required to be done.



THE APPENDIX. (*a*)

The question to be here briefly noticed is, whether 'Euclid's Elements,' having regard to the apparent intention of the author (or authors) of that work, should be considered a treatise on one peculiar division of knowledge (one science) only—that is to say, a treatise on the inter-relation of the subjects of the science and of the laws which govern and belong to that peculiar (so-called) science of geometry; or whether the work should be considered, in regard to its purpose, a practical treatise on applied reasoning—teaching, by illustration, the correct mode of building-up (compounding) science from its elements.

We will briefly examine two important propositions of those belonging to the 'Elements' as to any evidence they may afford in this respect.

The first prop. of the third book: 'To find the centre of a circle.' It is at once evident that this prop., because of the use made by Euclid of the circle, must be considered, if regarded as one of *the Elements* of a peculiar science (geometry), as a prop. of great relative importance. The plan of the book, assuming the purpose of that plan to be a treatise on the science only, would call for a solution of this prop. in such a form as to constitute it a primary or fundamental proposition, upon which secondary propositions and corollaries could be based and shown to be directly dependant. One or more definitions or axioms might obviously be provided to furnish any necessary support for a concise and comprehensive solution. At least, in regard to such supposed purpose, a direct character for the solution would suggest itself as almost imperative. As it stands, the character of the solution is indirect, and may be termed negative; it gets constructively at the answer to the requisition by a very simple operation, but, as a reasonable and subjective proceeding, this operation is only supported by the negative results of an exhaustive process. The question naturally suggests itself whether the construction has been, in the first place, merely the result of a fortunate guess. If it is based upon and has been compounded from elements previously verified, why is it not shown to be so? As it stands, we cannot take the result, and by inverting the order of the reasoning, get at the construction as subjective. No reason for performing the (given) constructive operation is apparent.

Again, the corollary is stated to be manifest; but, it is certainly not manifest as a corollary to the negative solution of the prop. If the corollary is manifest, it is manifest independently (in itself), and the solution of the prop. might be very well based upon the corollary. By exhibiting the fact stated in the corollary, in the form of a demonstrated theorem, the problem (Prop. I.) might be directly and positively solved in a few lines, and Prop. III. (Theorem) might become unnecessary.

Now, if we suppose the primary purpose of the work was to illustrate the applied method, and that the particular science (chosen for the purpose of illustration) was considered quite subordinate to that purpose, it is much easier to understand how such a form of solution might suggest itself as desirable in this and in some other similar intances. The negative exhaustive process illustrated in this prop. is undoubtedly in some cases of much value, and it may be considered, in some sense, a distinct kind of reasoning; hence, having regard to the primary purpose, it might be thought desirable to exhibit it as illustrating a proposition which from the fundamental character and importance of its subject would be more likely to attract attention and interest the mind.

We cannot consider this as furnishing an altogether satisfactory explanation of the form here given to the solution, but it seems to be much more intelligible when so considered, viz., as primarily intended to illustrate the applied method of reasoning.

The second proposition of the 12th book is a similar and still more striking instance, which appears to strengthen and confirm the explanation just suggested in the case of Prop. I. Book III. This proposition (Prop. II. Book XII.) 'Circles are to one another as the squares of their diameters,' is also evidently a proposition of a primary and important character because defining the nature of a fundamental relation between circles, and therefore here again we should expect a direct, positive, and simple treatment of the case, and, on the contrary, we again find the indirect negative treatment exhibited in a long solution of considerable complexity; the only characteristic difference between the treatment in this case and that of Prop. I. Book III., is that in this, the construction, like the solution to which it belongs, is complex and only indirectly related to the proposition, whereas, in the former instance, the construction is simple and directly related to its proposition. But the particular mode of the reasoning, for the illustration of which this proposition appears to have been selected, is in itself of a very refined and instructive character. In the kindred science of 'Number and Quantity,' calculations of great importance are derived from and based upon the quantitive equivalent of this proposition: for example —'Legendre's Numerical and Trigonometrical Propositions' concerning the quantitive relations of polygons and the perimeters of circles.* It may be also useful to point out that this is one of the propositions in which Euclid defines, by illustration, the use of the hypothesis. In this instance, the hypothesis is assumed to be true, and is then made to temporarily occupy the place of, and serve as a fact, so that combination can be carried forward and the results tested by comparison. The terms in which this proposition is stated require particular notice, in regard to the sense in which the expression 'square of the diameter' is used; for, the very same expression is used in the kindred science of 'Number and Quantity,' and is used therein in an essentially different sense. Since, therefore, we are here, almost on the border land which connects these two sciences, there is much danger of the expression used in the sense belonging to the one science being mistaken for the same expression used in the sense belonging to the other science. The proposition under consideration is thus stated in the Elements of Euclid, 'Circles are to each other as the squares of their diameters.' The expression 'square of the diameter' here means (as used by Euclid) a square of which the diameter of the circle is one of the four equal sides. The statement would therefore have the same (equivalent) meaning if the word 'square' be left out and we read simply 'circles are to each other as their diameters.' To demonstrate the one statement is to demonstrate the other; because if two magnitudes be proportional one to the other, equimultiples of those magnitudes are proportional in the same ratio; and, if similar fractions of those magnitudes be taken from them respectively, the remainders are likewise proportional in the same ratio. But in the applied science of 'Number and Quantity,' the same expression, 'square of the diameter,' has a different meaning.

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* We may remark here the confusion liable to arise from using without qualification, the same expression 'geometrical' which is applied to the propositions of the science of Form and Magnitude, for those of the science of Number and Quantity. To these last some of Legendre's propositions belong exclusively, whilst others are of a hybrid character. It is true these sciences have a common boundary where they approach each other very nearly, but it is a stumbling-block in the way of the student to find them thus mixed together without indication as to their distinct and different characters. By calling the one Magnitudinal and the other Numerical or Quantitive Geometry, this objection would be obviated.

To correctly appreciate the nature of the difference it is necessary to observe that all quantity of magnitude is relative. It has always reference to a standard of comparison. A seeming exception to this is the unit of magnitude, but here the quantity, is itself equivalent to its standard of comparison. In trigonometry 'square of the diameter' means a quantity of magnitude represented by the diam eter taken as many times as there are units contained in that quantity of magnitude (diameter). An example will at once make the character of this difference quite obvious. Let the two diameters be proportional one to the other in the ratio of *four* to *two*—then the square of their diameters in the sense intended by Euclid will be—*four* taken *four* times, and *two* taken *four* times; that is, *sixteen* to *eight*, magnitudes proportional in the same ratio as before. But in the numerical or quantitive sense we get—*four* multiplied by *four* gives sixteen (4 × 4 = 16); and—*two* multiplied by *two* gives *four* (2 × 2 = 4). The quantitive proportion becomes therefore 16 : 4, instead of 16 : 8;—the ratio of the proportion is no longer the same as before.

COPIES OF THE DOCUMENTS REFERRED TO IN THE PREFACE.

MONTREAL, 20th December, 1872.

To G. B. AIRY, ESQ.,

President of the Royal Society, London.

SIR,—I have the honour herewith to send you papers containing the solution to that geometrical problem (commonly or vulgarly known as 'Squaring the Circle'), which requires 'that a straight line shall be drawn equal in length to a given arc of a circle of definite magnitude,' or 'that a definite arc belonging to a circle of definite magnitude shall be described equal in length to a given straight line,' and which requires that the demonstration (proving that every condition of the requisition has been fulfilled,) shall be in strict accordance with the system of reasoning laid down in that work generally known, and recognized by mathematicians, as the 'Elements of Euclid,' or 'Euclid's Geometry.'

The papers are submitted to you for the purpose of their being examined by those whose mathematical knowledge entitles them to express an authoritative judgment as to the merits of the case thus laid before the Royal Society; and, as the subject is one of great public importance, I have to request that as soon as the necessary examination has been made (*i. e.*, within reasonable time and without unnecessary delay,) the correctness of my demonstration and the result of that demonstration shall be publicly acknowledged and admitted as established fact; or, otherwise, if disputed—then I require, which I submit that I have an unquestionable right to do, that the objection or objections shall be made strictly in accordance with the same system of reasoning—namely, that laid down and illustrated by Euclid,—in order that I may have the opportunity to meet such objections and, if I am able, to disprove their validity, and so to put myself in a position to insist upon my demonstration being publicly acknowledged as sound and true.

Yours respectfully,

JOHN HARRIS.

The letter accompanying the papers sent to McGill College, was of the same tenor, and in form substantially the same, as the foregoing.

ROYAL OBSERVATORY, GREENWICH,
LONDON, S.E.,
February 4, 1873.

SIR,

I have to acknowledge your letter of December 20, addressed to me as President of the Royal Society, (which has reached me through the hands of John Marshall, Esq.,) inclosing papers which are supposed to contain an investigation on trustworthy principles of the problem of "Squaring the Circle."

You are perhaps aware that the Foreign Academies, in general, by express statutes, refuse to receive communications on this subject. I do not know that the Royal Society of London is prevented by statute from receiving them, but I know that its practice is unvarying; and, so far as my private judgment is concerned, I approve of that practice. I must, therefore, decline to present the papers to the Royal Society.

I cannot myself give any time to their examination; nor should I think it right to force them on the attention of others. I have therefore thought it best at once to return them entire to Mr. Marshall.

I am, Sir,
Your obedient servant,
(Signed,) G. B. AIRY.

JNO. HARRIS, ESQ.

MONTREAL, 21st December, 1872.

To MR. JOHN HARRIS,

SIR,

I herewith return your papers relating to the problem known as Squaring the Circle. I have not received from any of the professors in McGill College any observations on the subject.

Your obedient servant,
(Signed,) CHAS. D. DAY.

THE CIRCLE

AND

STRAIGHT LINE.

PART SECOND

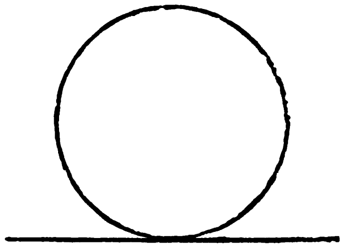
THE CIRCLE

AND

STRAIGHT LINE

PART SECOND.

THE CONSTRUCTION OF THE CIRCLE
IN RELATION TO
THE SCIENCE OF QUANTITY AND NUMBER.



BY
JOHN HARRIS.

MONTREAL:
JOHN LOVELL, ST. NICHOLAS STREET.

FEBRUARY, 1874.

Revised, Corrected and Reprinted,
October, 1875.

Entered according to Act of Parliament in the year one thousand eight hundred and seventy-five, by JOHN HARRIS, in the office of the Minister of Agriculture and Statistics at Ottawa.

PREFACE TO PART SECOND.

IN commending to the reader's attention this, the Second Part of the Circle and Straight Line, it may be of service to explain that the immediate purpose of this part of the work is three-fold. Of these..the first is, to furnish collateral and independant demonstration of the fundamental Theorem, by Quantitive Geometry.

The second is, to explain and illustrate the method of differential linear analysis.

The third is, to prepare the ground for the introduction of the study of Cyclometry as a division of Science.

The process of the radial duplication of the bisected arc is an analytical method, having an interest and value independently of the present investigation. The primary explanation and illustration of the method will be found in the Appendix to this Part of the work. Examination and application thereof as an instrument of analysis, will be contained in Part Third.

The objection to Legendre's mode of applying the inscribed and circumscribed polygon process to obtain the quantitive quadrature of the circle, may be stated in two ways, which, although essentially the same and arriving at the same conclusion, appear to differ much as modes of explanation. There are in fact two fallacies involved, so connected that either of them may be supposed to have been antecedent or consequent to the other.

The second mode of stating the objection belongs to the relation of the tangent to the arc of the circle, and in arranging the subjects of this work, has found its place in Part Third. In order to bring the merits of the objection under the consideration of the reader in one place, without confusion, we have now included this second mode of proposing and explaining the objection, as a supplement to this division (Part Second) of the work.

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THE CIRCLE AND STRAIGHT LINE.

'Prove *all* things; hold fast *that* which is *good.*'
St. Paul.

PART SECOND.

The Construction of the Circle.

Having in the preceding exposition of the subject demonstrated 'the true relation of the circle to the straight line' in accordance with Euclid's methodical application of the inductive system of reasoning, we will now proceed to investigate and illustrate the general relationship thereby established, in order—to make, in some measure, apparent the structural completeness and perfection of the circle as a reality,—to show the circumstantial necessity for that particular inter-relationship of the parts which has been now demonstrated to be actually existent, and thereby—to make manifest the great importance of the circle as one of the fundamental facts belonging to the Plan of Creation.

Let us take, in the first place, the fundamental construction of Fig. 1 & 2 of Part First, in which it was demonstrated that the sine of the arc contains a certain number of equal divisional parts, each of them equal to the diff. of the sine and the arc-length, and develope that construction as follows:—

Figs. 10 & 11. CONSTRUCTION. (Fig. 11 is a repetition of Fig. 1. on a smaller scale, similar letters denoting similar parts.) Produce the radius *A.B.* through *A.* and through *B.* and make A.B., (Fig. 10,) the production of *A.B.*, ten times the length of *A.B.* With centre A. and radius A.B., describe the quadrant B.F; bisect the quadrant at M.

and divide the arc M.F. into ten equal parts at the points of equal division *'a. 'b. 'c. 'd. 'e. 'f. 'g. 'h. 'i.* Complete the greater figure similarly to the lesser figure.

*Illustration of the fact that the difference between the chord and arc-length of the quadrant is an aliquot part of the chord, and of the arc-length.** (*Corollary to Fund. Demonstration, Part First, page* 31.)

Because R.M. in the greater figure equals *S.R.* and *R.M.*, of the smaller figure, taken together (to wit, *S.M.*), it follows that however small the magnitude of a circle may be, the arc-length and the chord of the quadrant of that circle are each divisible into a certain number of equal parts, each of which parts is necessarily equal to the difference of the chord and arc, and each of which parts is an aliquot part of the chord and of the arc-length of the quadrant of any other circle of which the magnitude is an equimultiple of the magnitude of the first circle, however great that other circle may be. Now it is to be particularly observed that, if R.M. of the greater figure were not equal to *S.R.* and *R.M.* of the lesser figure taken together, this would be no longer true. In that case either *S.R.* or *R.M.* of the lesser figure might be evenly divided, but the equal divisional parts would not be aliquot parts of the compound parts belonging to the circle of greater magnitude. If the relative magnitude of the greater circle were an equimultiple of that of the lesser circle .. R.M. of the greater, would contain an equimultiple of *R.M.* of the lesser; but, then, S.M. of the same greater circle, could not also contain an equimultiple of *R.M.* of the lesser, nor of *S.R.* of the lesser, neither could S.R. of the greater, contain an equimultiple of *R.M.* of the lesser.

The general theorem illustrated by this construction may be thus stated. The difference between the chord

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* Evidently, in this connection, the relation of the half-chord to the half-quadrant is equivalent to the relation of the chord to the quadrant.

and arc-length of the quadrant of a circle, is an aliquot part of the circle; and if a second circle be so related to the first that the magnitude of the first is an aliquot part of the magnitude of the second, then is the difference of the chord and arc-length of the quadrant of the first, also, an aliquot part of the second circle.

The relation of the Sine to the arc-length of the Octant, the basis of the Science of Number and Quantity.

In Theorem (Fig. 7), of Part First, we have made known and have utilized the definite decimal relation of the versed sine to the arc-length and sine, of the octant. We will now exhibit this relationship in the form of a special theorem, as an independent fact.

PROP. F.—THEOREM.

If the radius of an octant be produced through the centre of the circle until it become equal to ten times the length of the versed sine of the octant, and with this production of the first radius an arc be described on the same tangential line as the octant, terminated by its secant drawn through the terminal extremity of the octant, the tangent to the arc of greater magnitude so described shall be equal to the arc-length of the octant.

Fig. 25, R.—Let. *B.S.* be the octant, *A* the centre of the circle, and *A.B.* the radius.

Produce the radius *A. B.* of the octant, through *A.* the centre of the circle and make the production *R. B.* ten times the length of the versed sine *N. B.* With centre *R.* and radius *R. B.* describe the arc *B. d.*, terminated by the secant *S.X.* drawn through the terminal extremity *S.* of the octant *B. S.* The tangent *B. X.* of the arc *B. d.* shall contain the arc-length of the octant *B. S.* (The demonstration to this theorem is contained in that already given to Prop. E.) Because the triangle *R. B. X.* is similar to the triangle *R. N. S.*, and the ratio of the rectangle *R. B. X.* of the greater, to the rectangle *R. N. S.* of the lesser, is ten to nine, therefore the ratio of the line *B. X.* to the line *N. S.*, is ten to nine. Now it has been demonstrated that the ratio of the arc-length of an octant, to the sine of that octant, is the ratio of ten to nine, and the line *N.S.* is the sine of the octant *B.S.* *Wherefore* the line *B. X.*, the tangent of the arc *B. d.* contains the arc-length of the octant *B. S.*

The application of the fact here shown in the construction of the circle will become more apparent by developing the figure.

Fig. 25, RR. (1).—The line N S. is divided into ten equal parts at the points of equal division, 1, 2, 3, 4, 5, 6, 7, 8, 9, and the line B.X. is also divided into ten equal parts, of which B.W. (equal to N.S.,) contains nine. Now, since the secant R.S.X., which intersects N.T. at S., intercepts B.E. at X., if a line be drawn from the point R. at the extremity of the produced radius, through the first divisional point (1) on the line N.S., the line so drawn will intercept B.X. also at the first divisional point thereof: for, each of the equal divisional parts of the line B.X. is greater by one-tenth of itself than each of the equal divisional parts of the line N.S., and the whole (produced) radius R.N.B. is greater, by one-tenth thereof, than the line R.N.; consequently, if a line, from the point R., be drawn through each of the successive points of equal division on the line N.S., the lines so drawn will respectively intercept each of the nine successive points of equal division of the line B.X.

It is evident, therefore, that however great or however extremely small may be the distance moved through by the radius R.B., between the points N. and S., on the line N.S., that quantity of distance will be increased by one-ninth of that quantity on the line B.X. intersected by the secant; so that, whatever fraction or part of the sine (N.S.) of the half-quadrant, is moved through by the radius, the produced radius describes

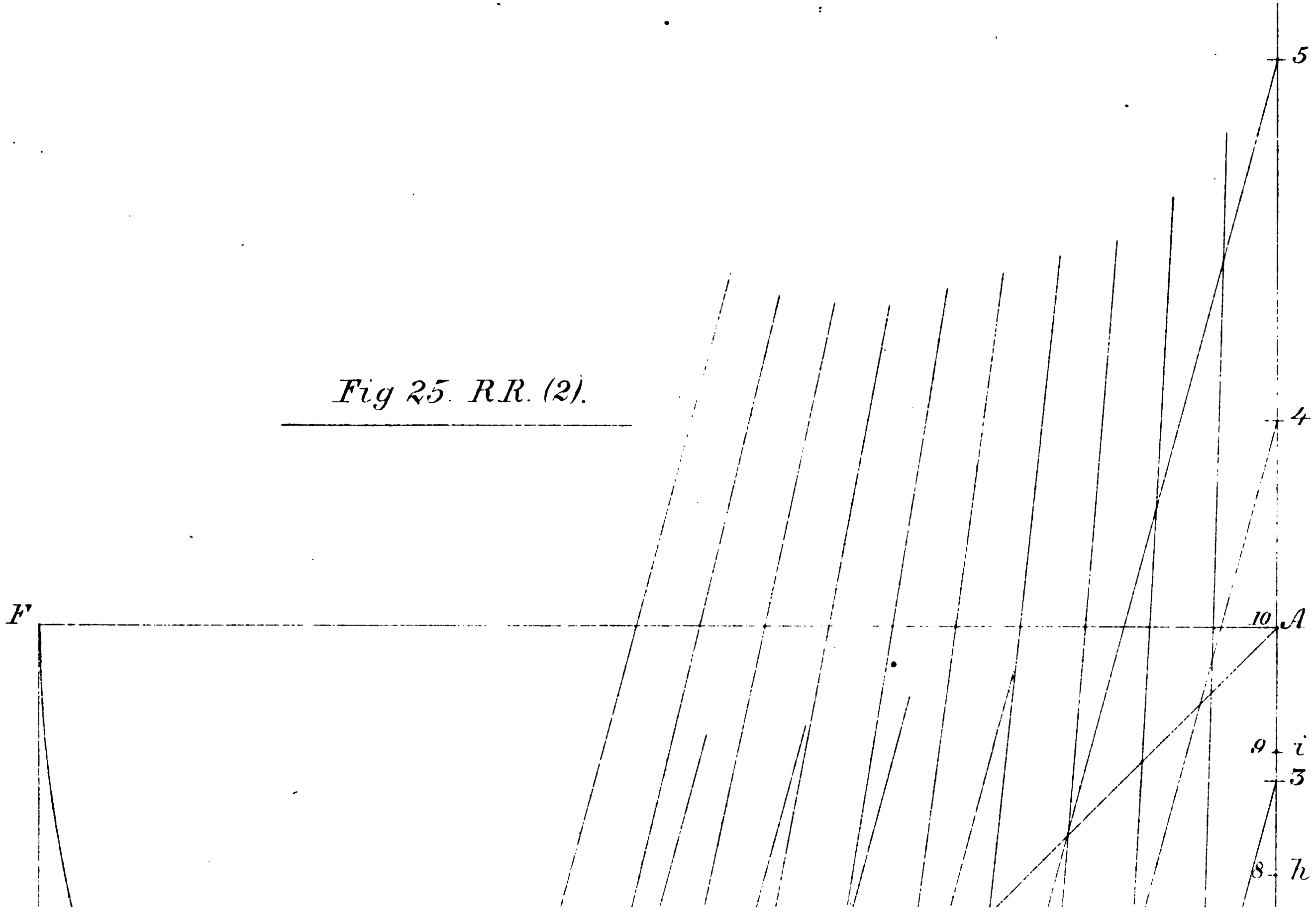
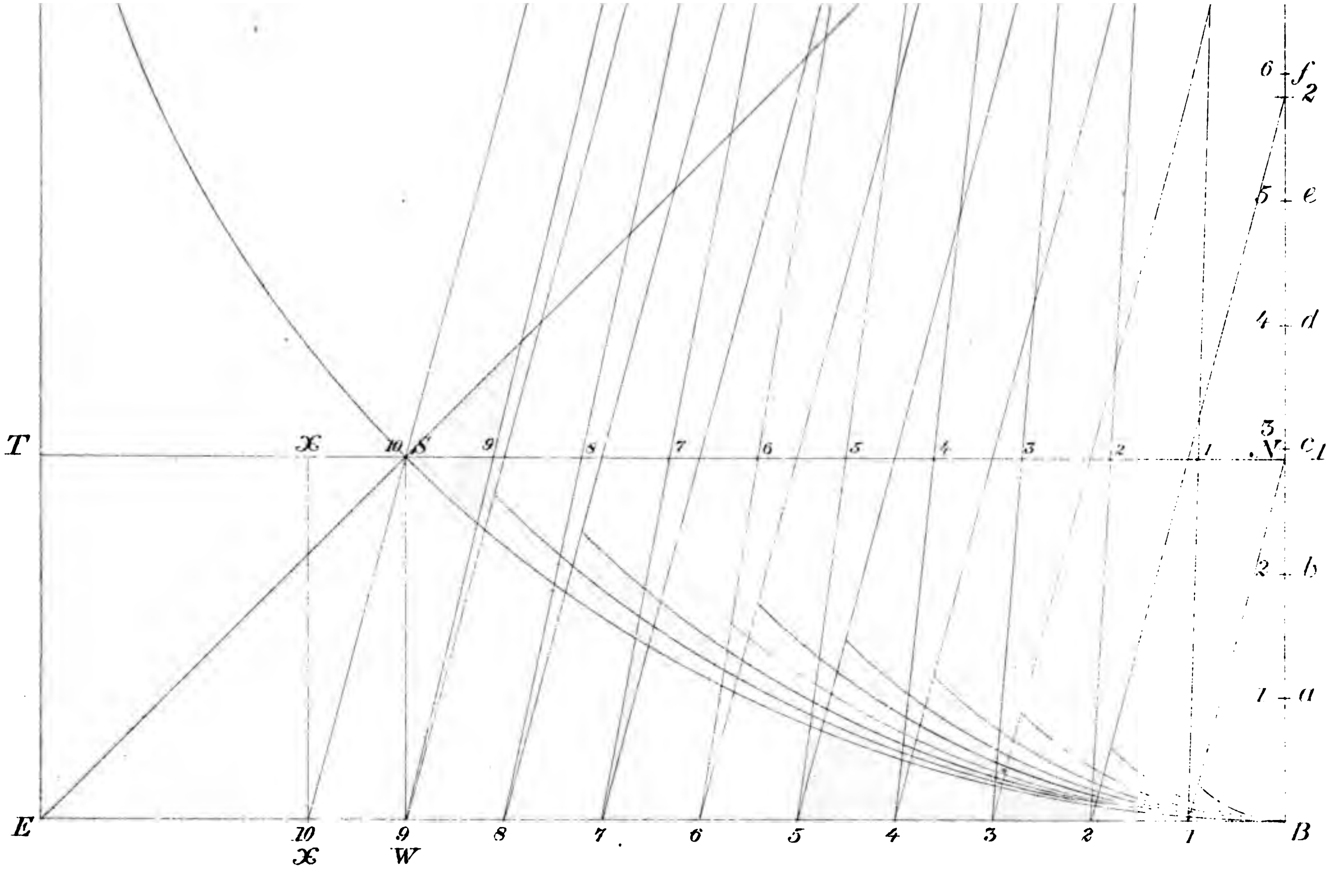


Fig 25. RR. (2).



T
E
B
X
S
N
c
6 f
5 e
4 d
2 b
1 a
X
W

on the tangential line (B.𝔛.) the corresponding (or proportional) part of the arc-length of the half-quadrant.

Again, if from the point N., of the line N.S, a line parallel to S.𝔛. be drawn to the line B.𝔛,. the line so drawn will intercept the first divisional point (1) of the line B.𝔛. And, if a point in the perpendicular R.B. be taken at twice the distance of the point N. from B., (*i.e.*, the second divisional point (2) on the line R.B.,) and from this point a line parallel to S.𝔛. be drawn to the line B.𝔛., the line so drawn will intercept the second divisional point (2) on the line B.𝔛.; and, similarly, lines parallel to S.𝔛. drawn from each of the successive divisional points on the radius R.B. will intercept the successive points of equal division of the line B.𝔛.

We now refer to the repetition of the figure at Fig. 25, R.R. (2). With the first decimal point on the perpendicular R.B. as a centre, and with the radius a.B., describe the first half-quadrant, (one-tenth the magnitude of the half-quadrant B.S.; similarly, with centre b, and radius b.B., describe the second half-quadrant (two-tenths the magnitude of B.S.,); with centre c, and radius c.B. describe the third half-quadrant (three-tenths the magnitude of B.S.,) and so on. The illustration furnished by the two figures may be appreciated without further minute explanation. It will suffice to observe that the sine of each successive half-quadrant is equal in length to the corresponding divisional part of the line (N.S.) above it, and that the arc-length of each successive half-quadrant equals in length the corresponding divisional part of the line(B.𝔛.) beneath it.

Magnitudinal and Quantitive Geometry.

Quantitive Geometry, as distinguished from magnitudinal geometry, may be explained as consisting in the application of numerical ratios to magnitudes, in order to determine their contents in relation to each other.

Instead of comparing together and determining the relations of magnitudes as wholes, each magnitude is resolved into the number of standard increments or components which it contains; and, by comparison of the number contained in the one with the number contained in the other, the quantitive relation is estimated and determined.

The great advantage in convenience and simplicity of the numerical method where the object is to estimate the difference of two or more magnitudes nearly but not quite equal, or, where the ratios are compounded of fractional quantities difficult to express as whole magnitudes, is obvious.

The differential method of analyzing the quantitive magnitudinal value of the compounding parts of a given line, by the application of numerical ratios, is based upon the axiom that..if a given definite magnitude—*i.e.*, a magnitude containing a known definite quantity of magnitude—be taken from a greater known definite magnitude, a third known definite magnitude remains.

The precise nature of this linear analytical process may be shown, and the mode of its application be illustrated, by the actual application of the process to the subject of the present investigation, thereby testing quantitively the truth and soundness of the magnitudinal result obtained in answer to the requisition of our fundamental theorem For this purpose we will, in the first place, repeat the second demonstration furnished in Part First to the fundamental theorem.

Proposition (in quantitive geometry.)

THEOREM.—*The quantity of length by which the length of the half-quadrant of a circle exceeds the length of the sine of the half-quadrant, is equal to the one-tenth part of the length contained in the half-quadrant.*

Fig. 12. (*a*)—*Construction.*—With centre *A*. and radius *A.B* describe the quadrant *B.S.C.*; draw *B.D.* and *C.D* and join *A.D.* bisecting the quadrant at *S.* Through *S.*, and at right angles to *A.B.* draw *R.S.T.*, intercepting *A.B.* at *R.*, and *C.D.* at *T.* Divide *R.S.* into nine equal parts at the points of equal division; from *S.T.*, at the distance *S.𝔛.* equal to one of the equal divisions of *R.S.*, draw *𝔛.𝔛.* perpendicular to *B.D.*, intercepting *B.D.* at *𝔛.* Divide the radius *A.B.* into ten equal parts at the points of equal division. With the point *S.* as a centre and with the radius *S.b.*, equal to one of the equal divisional parts of the greater radius *A.B.*, describe the small quadrant *b.s.t.*, bisected by the line *A.D.* at the point *s.* Draw the tangent *b.d.* and the sine *S.r* of the small arc *b.s.* Join *t.d.*, and from *s.* draw the perpendicular *s.s̄.* intercepting the line R.T. at *s̄.*

Specification.—Let *B.S.* be the half-quadrant of which *S.R.* is the sine, and let *S.𝔛.* indicate the diff. in length between *B.S.* and *S.R.*;—*S.𝔛.* shall equal one of the divisional parts of the arc *B.S.*, if *B.S.* be divided into ten equal parts.

Demonstration.—Because the radius *S.b.* is the one-tenth of the radius *A.B.*, the sine *S.s.*, of the lesser arc *b.t.*, is one-tenth of *R.S.* the sine of the greater arc. But *S.𝔛.* the diff. of the arc-length and sine of the greater arc, is ten times greater than the similar diff. *s.𝔛.* of the arc-length and sine of the lesser arc; therefore *S.s.* + *s.𝔛.* that is *S.𝔛.*, is the one-tenth of *R.𝔛.* Now *R.𝔛.* is the arc-length of *B.S.*; and *S.𝔛.* is the diff. between that arc-length and the sine of the same arc. Wherefore the theorem has been demonstrated.

Scholium. Since the radius of the small arc is one-tenth the radius of the greater arc; the arc, the sine and the tangent of the small arc are respectively one-tenth of the arc, of the sine, and of the tangent, of the greater arc; and therefore, also, the *difference* of the *sine and arc* on the small scale is one-tenth of the *difference* of the *sine and arc* on the larger scale (*i.e.*—*𝔛.s.* is the one-tenth of *𝒳.S.*) But *s.t.* the *difference* of the *sine and tangent* on the small scale is manifestly one-tenth of *S.T.* on the large scale, and *𝒳.t.* must be also necessarily one-tenth of *𝒳.T.* It will follow, if the small arc *b.s.* is straightened upon or rolled upon the line *b.d.*, that the point of contact must fall upon the line *𝒳.𝒳.*

The quantitive analysis of this demonstration, taking the numerical ratios in units of radius, is:—

The greater radius *A.B.* = 10.
The sine *R.S.*, of the greater octant *B.S.*, $=\sqrt{50}=$ 7·071068
By demonstration, the diff. ‘of the sine
and arc-length’ equals (7·071068 ÷ 9) = *S.𝒳* = 0·785674
And (*R.S.* the sine + *S.𝒳.* the diff.) equals *R.𝒳* = 7·856742

The lesser radius *S.b.* = (*A.B.* ÷ 10) = 1.
The sine *S.s.*, of the lesser octant *b.S.*, $=\sqrt{\tfrac{1}{2}}=$ 0·707107
Diff. of the sine and arc-length of the lesser *s.𝒳* = 0·078567
And *S.s.* + *s.𝒳.* equals the arc-length in the lesser fig. } *S.𝒳* = 0·785674

Therefore, since by the demonstration to the theorem, *S.𝒳.* the diff., is one-tenth of *R.𝒳.* the arc-length, the numerical result is quite in harmony with and verifies the magnitudinal result.

In the demonstration to the fundamental theorem, in Part First, we tested the positive direct demonstration by negative supposition. Now in similarly testing the quantitive demonstration negatively, we may also, at the same time, test positively the quantitive value assigned at the present time to the arc-length of the octant, by supposing that the latter quantity is, or may be, the actual quantitive value.

Quantitive and Numerical Illustration (demonstration) *of the fact* that the quantity obtained by analytical methods, which is now supposed to represent the ratio of the circumf. of a circle to the diameter is erroneous.

Scholium. The quantity under examination, stated as the ratio of the circle to a unit of diameter, equals 3·14159; therefore, taking the radius = 10. the arc of 45 degrees = 7.85397. . . .

By Fig. 12. Taking the radius A.B. = 10.

And assuming the arc-length to be *as stated*, then

R.$\mathcal{X}$. = 7·85397 R.T. = A.B. = 10

R.S. = $\sqrt{50}$ = 7·071067 And therefore:—

S.$\mathcal{X}$ = (R.$\mathcal{X}$. − R.S.) = ·78291

$$\text{Again, S. s.} = \frac{\text{R.S.}}{10.} = {\cdot}70710678$$

$$\text{And, s.}\mathcal{X}. = \frac{\text{S.}\mathcal{X}.}{10.} = {\cdot}078291$$

Therefore S.$\mathcal{X}$. = (S.s. + s.$\mathcal{X}$.) = ·78539. . .

But it has been shown that S.$\mathcal{X}$. = ·78291, and therefore the same line has (appears to have) two different lengths which is impossible.

And again, further:—

$$\text{R.s.} = \text{R.S.} + \frac{\text{R.S.}}{10} = \text{S.s.} \times 11 = 7{\cdot}7781745,$$

By the assumption; *R.$\mathcal{X}$.* = 7·85397

(R.$\mathcal{X}$ − R.*s*.) × 10 = (s.$\mathcal{X}$.×10) = S.$\mathcal{X}$.

(7·85397 − 7·778174.) = ·075796. . .

Therefore *S.$\mathcal{X}$.* = ·75796. . .

But it has been already shown that S.$\mathcal{X}$. equals ·78539, and also equals ·78291—and therefore the same line S.$\mathcal{X}$. is (apparently) demonstrated to have three different lengths, which is absurd.

Again by the assumption; R.$\mathcal{X}$. = 7·85397.

$$\text{But S.}\mathcal{X}. = \frac{\text{R.}\mathcal{X}.}{10} = {\cdot}785397.$$

R.$\mathcal{X}$. = (S.$\mathcal{X}$. + S.R.) = (·785397 + 7·0710678) = 7·85646.

Therefore R.$\mathcal{X}$. has (appears to have) two different lengths which is impossible. Wherefore the assumption is erroneous.

Moreover, by demonst., *R.𝔛.* = 7·856742

S.T. (the greater) diff. of sine and tangent = 2·928932
S.𝔛. (the greater) diff. of sine and arc = ·785674
𝔛.T. = 2·143258
s.t. (the lesser) diff. of sine & tangent = ·292893
s.𝔛. (the lesser) diff. of sine and arc = ·078568
𝔛.t. = 0·214325
(*S.T.* – *S.t.*) = *t.T.* = 1·928933
And *R.T.* – (*R.S.* + *S.t.*) = *t.T.*
i. e., (10 – 8·071068.) = *t.T.* = 1·928933

Now if we again suppose *R. 𝔛.* = 7·85397
Then *S.T.* (the greater) diff. of sine and tangent = 2·92893
S.𝔛. (the greater) diff. of sine and arc = 0·78291
𝔛.T. = 2·14602
s.t. (the lesser) diff. of sine & tangent ·292893
s.𝔛. (the lesser) diff. *s*f sine and arc = 078291
𝔛.t. = 0·21460
And . . . *t.*T. equals . . . . 1·93142
But, since *R.T.* — (*R.S.* + *S.t.*) = *t.T.*
Therefore (10—8·071067) = *t.T.*
And *t.T.* must equal . . . . 1·92893
Because *S.t.* = (*R.T.* ÷ 10) = 1. And it is quite certain that
R.S. = √50 = 7·071068.

Development of the Construction. Fig. 12. (*b.*) From the point *𝔛.* on the line *B.D.*, at the distance *𝔛.V.* equal to *𝔛.D.*, take the point *V.* From *V.* at right angles to *S.D.* draw *V.F.*, intersecting *S.D.* on the line *𝔛.𝔛*, and intersecting the line *b.d.* at *v.*

Illustration —Now, if a second arc similar and equal to the greater arc *B.S.C.* be described in such wise that the point *V.* become the extremity of the secant, and the production of *V.F.* be the secant, and the production of

V.D. be the tangent, of the second greater arc, it is manifest that the second arc, so described, must, necessarily, intersect the first greater arc on a production of the line *𝔛𝔛.* Now *S.d.*, the secant of the small arc, is included in the secant *A. D.* of the first greater arc, and likewise, if a second small arc be described similarly related to the second greater arc, then will the secant of the second lesser arc be included in the (production of the) line *V.F.* the secant of the second greater arc; therefore since *x.d.* is one tenth of *𝔛.D.*, *x.v.* must necessarily be the one tenth of *𝔛.V.* It becomes evident accordingly that the point *x.* of the lesser arc necessarily falls upon the line *𝔛.𝔛.*

Fig. 12. (*b.*) may be further developed by completing the double figure. Produce *𝔛.T.* through *T.* and on that production make (*R.*) *𝔛.* equal to *R.𝔛.* Produce *A.C.* through *C.* and produce *B.D.* through *D.* and make *C.* (*A.*) and *D.* (*B.*) each equal to *T.* (*R.*) on the line *R.* (*R*). With centre (*A*) and radius (*A.*) (*B.*) describe the arc (*B.*) (*C.*) intersecting the arc *B. C.* in the line *𝔛.𝔛.* With the centre *F.* (on the line *R.* (*R.*) and radius *F.z.* equal to *S.b.* equal to (*A.B.* divided by 10), describe the arc *z.y.* Join *z.d.b.* Draw *n.m.* the sine of the arc. Join *y.v.* and produce *F.z.* through *z.* intercepting the line *B.*(*B.*) at *Q.*

The analytical value of the construction thus developed may be understood by considering that the line *R.* (*R.*) contains the sines *R.S.* and (*R.*)*F.* of both the arcs together with the space *S. F.* And *R.*(*R.*) also contains the tangents *R.T.* and (*T.*) (*R.*) of both the arcs, *less* *T.* (*T.*) Again, because the radius *F.z.* is equal to the radius *s.b.*, and because *F.z.* has the same relation to the arc (*B.*)(*C.*) that *S.b.* has to *B.C.* therefore *y.x.* is equal to *t.x.*, &c., &c.

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* Because *R.x*=*R.S.* divided by 9, × 10.

Example. The radius $A.B. = 10$.

$R.\,(R.) = 2\,R.T. - 2\,\mathscr{X}.T.$ equals..

$$20 - 4{\cdot}28652 = 15{\cdot}7134$$

$R.\,(R.) = 2\,R.S. + 2\,S.\mathscr{X}.$ equals..

$$14{\cdot}14214 + 1{\cdot}57134 = 15{\cdot}7134$$

$R.\,(R.) = 2\,R.T. + 2\,S.\mathscr{X}. - 2\,S.T.$ equals..

$$20 + 1{\cdot}57134 - 5{\cdot}85776 = 15{\cdot}7134$$

$R.\,(R.) = R.\mathscr{X} + (R.)\mathscr{X}.$ Which equals

$$\left.\begin{array}{l} R.\,S. + \dfrac{R.S.}{10} + \dfrac{R.S.}{100} + \dfrac{R.S.}{1000} + \&c. = 7{\cdot}S5674 \\ (R.)F. + \dfrac{(R.F.}{10} + \dfrac{(R.F.}{100} + \dfrac{(R.F.}{1000} + \&c. = 7{\cdot}85674 \end{array}\right\} .15{\cdot}7134$$

Illustration by the construction. Because the line $x.x.$ belonging to the small arc is included in the line $\mathscr{X}.\mathscr{X}.$ of the greater, the line $S.\mathscr{X}.$ is equal to $S.s.$ added to $s.x.$ (i.e. the *diff.* of the *sine and arc* on the large scale is equal to the sine of the arc added to the *diff.* of the *sine and arc* on the small scale.

Quantitive and Numerical Illustrations, to Fig. 12.

$R.T. = A.B. = 10{\cdot}000000$

$R.S. = \sqrt{50} = 7{\cdot}071068$

$A.D. = A.S. + S.D. = 14{\cdot}142135$

$S.D. = A.D. - A.S. = 4{\cdot}142135$

$S.T. = R.T. - R.S. = 2{\cdot}928932$

$R.s. = R.S. + \dfrac{R.S.}{10} = 7{\cdot}778174$

$$R.x. = \frac{R.S.}{9} \times 10 = 7{\cdot}856742$$

Because $S.s. = \dfrac{S.R.}{10}$ And $S.\,t. = \dfrac{S.T.}{10}$

And $S.s + s.t. = \dfrac{R.T.}{10}$ And $R.T. = R.S + S.T$,

Therefore (1) $R.s. + s.t. = R.t.$

(2) $R.T. - 9\,S.t. + R.S. = R.t.$

(3) $R.S. + S.t. = R.t.$

And

(1) 7·7781748 + ·292893 = 8·0710678

(2) 10–9 + 7·0710678 = 8·0710678

(3) 7·0710678 + 1·0000 = 8·0710678

$$\text{Because S. s.} = \frac{\text{R.S.}}{10} \quad \text{Therefore R.s.} = \text{R.S.} + \frac{\text{R.S.}}{10}$$

and consequently if the scale be again reduced to one tenth of the lesser figure, and then again reduced to one tenth of the last, and so on 'ad infinitum', R.$\mathfrak{X}$. must evidently include the (sum of the one-tenths) one-tenth of each and every of all the sines; (i. e., the sum of the one-tenths of the sines of all the figures from the greatest (R.S.) to the least imaginable). Therefore R.$\mathfrak{X}$. equals

$$\text{R.S.} + \frac{\text{R.S.}}{10} + \frac{\text{R.S.}}{100} + \frac{\text{R.S.}}{1000} + \frac{\text{R.S.}}{10000} + \text{\&c., ad inf.,}$$

7·0710678
70710678
70710678
70710678
70710678
70710678
&c., &c., &c., &c., ad infinitum.

7·856742, &c., &c.

Quantitive Illustration bg Fig. 12 (*a*), *of the fact that x. falls on the line* $\mathfrak{X}.\mathfrak{X}$.

$$\text{Because S.s.} = \frac{\text{R.S.}}{10} \quad \text{Therefore S.s.} = \frac{\text{S.R.} + \text{S.s.}}{11}$$

$$\cdot70710678 = \frac{7\cdot0710678}{10} = \frac{7\cdot0710678}{11} + \frac{\cdot70710678}{11}$$

And S.s.:S.x. : : S.R:R.$\mathfrak{X}$. Therefore

$$\text{S.}x. = (\text{S.s.} + \frac{\text{S.s.}}{9}) = (\cdot70710678 + \cdot07856742) = \cdot7856742$$

$$\text{R.}\mathfrak{X}. = \text{S.}\mathfrak{X}. + \text{R.S.} = (\cdot7856842 + 7\cdot0710678) = 7\cdot856742$$

$$\text{S.s.} = \frac{7\cdot77817458}{11} = \cdot70710678..$$

But S.s. : S.x. : : S.R. : R$\mathcal{X}$.

$$\text{S.}\,s. = \frac{\text{S.R.}}{10} \quad \text{R. s.} = (\text{S.R.} + \text{S.}s.) = 7{\cdot}778174$$

$$\text{S.}\,\mathcal{X}. = (\text{S.}s. + \frac{\text{S.}\,s.}{9}) = {\cdot}7856742 = \frac{\text{R.}\mathcal{X}.}{10}$$

$$\text{S.}\,\mathcal{X}. \times 10 = 7{\cdot}85674 = \text{R.}\,\mathcal{X}. = (\text{S.}\,\mathcal{X}. + \text{S.R.})$$

$$s.\,x. = \frac{\text{S.}\,x.}{10} \quad \text{And}(\text{R.}s. + s.\mathcal{X}.) = R.\mathcal{X}.$$

That is, $(7{\cdot}778174 + {\cdot}078567) = 7{\cdot}856742 = \text{R.}x.$

And again $\mathcal{X}$.T. = (R.T. – R.$\mathcal{X}$.) = 2·143258

$$\mathcal{X}.\text{t.} = (\text{S.t.} - \text{S.}\mathcal{X}.) = {\cdot}2143258 \text{ (i.e. } \mathcal{X}\text{t.} = \frac{\mathcal{X}.\text{T.}}{10}\text{)}$$

Because the lesser (right-angled) triangle S.W.D. is similar to the great triangle A.R.S., therefore

$$\frac{\text{S.W.}}{9} : \text{S.D.} : : \frac{\text{R.S.}}{9} : \text{A.S.}$$

$$\text{And } \frac{\text{S.W.}}{9} \times \frac{\text{A.S.}}{\text{S.D.}} = \frac{\text{S.R.}}{9}$$

$$\text{That is, } \frac{2{\cdot}928932}{9} \times \frac{10}{4{\cdot}14213} = \frac{3{\cdot}25437}{4{\cdot}14213} = {\cdot}7856742$$

Figure 10 and 11.

Quantitive and Numerical Illustrations to the construction.

(By the construction :—A.B. the radius of the greater figure (Fig.10), equals *A.B.* the radius of the lesser figure (Fig.11) multiplied by ten.)

By the established trigonometrical relationship of the parts :

	In the greater Fig. 10.	In the lesser Fig. 11.
The radius	A.B. = 10·000000	*A.B.* = 1·0000000
The sine √50	S.R. = 7·071068	*S.R.* = 0·7071068
The diff of the arc and sine $\frac{\text{S.R.}}{9}$	= M.R. = 0·785674	*R.M.* = 0·0785674

The arc length M.F. or S.M. = 7·85674

The arc length *M.F.* or *S.M.* = 0·785674

Therefore:

$$\text{R.M.} = \frac{\text{S.M.}}{10} = 0{\cdot}785674 \qquad S.M. = 0{\cdot}785674$$

Fig 13.

Construction.—Describe the quadrant *B.C.* bisected by the line *A.D.* in the point *S.* Draw the sine *R.S.* and the tangent *B.D.* of the arc *B.S.* Join *A.C.* and *D.C.*, and produce *R.S.*, through *S.*, intercepting *D.C.* at *T.* Divide the radius *A.B.* into ten equal parts, at the points of equal division. With centre *S.* and radius *S.b.*, describe the quadrant *b.t.* bisected by *S.D.* in the point *s.*; draw the sine *s.r.* and the tangent *b.d.*, of the arc *b.s.*

Produce the line *B.D.* through *D.*, and make *𝔛.F.* equal to *𝔛.B.*; produce *A.C.* through *C.*, and from *F.* draw *F.E.*, perpendicular to *B.F.*, and intercepting the production of *A.C.* in the point *E.* From the point *n.*, where *A.D.* intersects *𝔛.𝔛.*, join *n.B.*, and from the same point join *n.F.* intersecting *C.D.* at *H.*, and from the same point draw also *n.I.E.* at right angles to *A.D.* intersecting *C.D.* at *I.*, and intercepting the point *E.* at the vertex of the angle *A.E.F.* With centre *I.* and radius *I.H.*, describe the quadrant *H.K.* bisected by *I.n.* in the point *e.* Draw the sine *G.e.* of the arc *H.e.*, and from *d.*, through *e.*, draw *d.t.* perpendicular to *R.T.* and intercepting *R.T.* at *t.* From *K.*, at the extremity of the quadrant *H.K.* draw *K.m.* perpendicular to *R.T.* and intercepting *R.T.* at *m.*

Scholium.—We have now three similar triangles. namely: *A.n.B.*, *S.n.b.*, *I.n.H.* Because *n.* is a point in the line *𝔛.𝔛.*, and *T.* is a point in the radius *I.H.*, and *R.* a point in the radius *A.B.*; the ratio of the radius *I.H.* to the radius *A.B.* is the same as the ratio of the line *𝔛.T.* to the line *𝔛.R.*, and also, the ratio of the radius *S.b.* to the radius *I.H.*, the same as the ratio of *S.𝔛.* to *T.𝔛.*

Illustration (*a*) *of the fact that the point x falls on the line 𝔛.𝔛.*

Because the radius *S.b.* is the one-tenth of the radius *A.B.*, therefore *s.x.* is the one-tenth of *S.𝔛.*, and *x.t.* the one-tenth of *𝔛.T.* But *S.t.* (*i.e.*, *S.x.* + *x.t.*) is the one-tenth of *R.T.*, and the ratio of *S.x.* to *x.t.* is the same as the ratio of *R.𝔛.* to *𝔛.T.* (because the ratios of similar arcs each to each, are directly as the ratios of the radii each to each, to which they respectively belong.) And the ratio of *R.S.* to *T.t.* is also the same as the ratio of *R.𝔛.* to *𝔛.T.*, therefore *x.T.* is equal * to *𝔛.T.* and *x.R.* equal to *𝔛.R.* Wherefore it becomes evident that the point *x.* is included in the line *𝔛.𝔛.*

By the same reasoning applied to the secant,

Because *S.b.* is the one-tenth of *A.B.*, therefore *S.d.* is the one-tenth of *A.D.*, and *S.s.*, the one-tenth of *R.S.*, and *s.d.* the one-tenth of *S.D.* But *s.n.* is, therefore, the one-tenth of *S.n.*, and *n.d.* the one-tenth of *n.D.* Consequently the point *n.*, which is a point in the line *x.x.*, is also included in the line *𝔛.𝔛.* Wherefore it clearly appears that the line *x x.* is the same line as *𝔛.𝔛.*

Demonstration (*a*) by the construction Fig. 13) to prop. B.) *That the diff. of the sine and arc-length of the half-quadrant contains one-tenth the quantity of length contained in the half-quadrant.*

$$\text{Because S.t.} = \frac{\text{R.T.}}{10}, \text{and, } s.\,\mathfrak{X}. = \frac{\text{S.}\mathfrak{X}.}{10}$$

$$(\text{R.T.} - \text{S.t.}) = \frac{9\ \text{R.T.}}{10} = (\text{R.S.} + \text{T.t.})$$

$$\text{And R.T.} = (\text{R.S.} + \text{T.t.}) + \text{S.t.} = (\text{R.S.} + \text{S.}x.) + (\text{T.t.} + \text{t.}x.)$$

$$\text{and, also, R.T.} = (\text{R.S.} + \text{T.t.} + \frac{\text{R.S.}}{9} + \frac{\text{T.t.}}{9})$$

---

* Because $x.T. = (x.t. \times 10)$ and $x.R = (x.r. \times 10.)$

Therefore S.𝔛. equals $\frac{\text{R.}\mathfrak{X}.}{10}$ And t.𝔛. equals $\frac{\text{T.}\mathfrak{X}.}{10}$

For let it be supposed possible that S.𝔛. may be a magnitude less than $\frac{\text{R.}\mathfrak{X}.}{10}$ then must t.𝔛. be a magnitude greater than $\frac{\text{T.}\mathfrak{X}.}{10}$ and *s*.𝔛. less than $\frac{S.\mathfrak{X}.}{10}$.

But R.T. is wholly compounded of R.𝔛. and T.𝔛. together. (Demonst. b), and R.T. contains R.S. and T.t. together with $\frac{\text{R.T.}}{10}$ (i.e. $\text{R.S.} + \text{T.t.} + \frac{\text{R.S.}}{9} + \frac{\text{T.t.}}{9}$) and $\frac{\text{R.T.}}{10} = (\frac{\text{R.S.}}{9} + \frac{\text{T.t.}}{9})$ consequently if S.𝔛. be any magnitude less than $\frac{\text{R.S.}}{9}$ then must the remaining magnitude, of which together with $\frac{\text{R.S.}}{9}$ the magnitude $\frac{\text{R.T.}}{10}$ is compounded, be greater than $\frac{\text{T.t.}}{9}$

Now T.t. is the sine of an arc similar to the arc of which R.S. is the sine, and therefore it is impossible that the ratio of t.𝔛., the diff. of the sine and arc length of H.e., to T.t., the sine of the arc H.e., can be greater than the ratio of S.𝔛., the diff. of the sine and arc-length of B.S., to R.S. the sine of the arc B.S. Wherefore it is demonstrated that S.𝔛., the diff. of the sine and arc-length of the half-quadrant B.S., is the one-tenth part of the arc-length contained in the half quadrant B.S. Q.E.D.

Demonst. (b.) *That the line R.T. is wholly compounded of the arc-length of B.S. together with the arc-length of H.e.; and that the same point x., the extremity of the arc-length of each of the two arcs—to wit, of the arc B.S. and the arc H.e., coincides with the point 𝔛.*

(If the arc-length of B.S. contained between R. and a point indicated by 𝔛. in the line R.T. be taken from R.T.

it is manifest that a magnitude will remain which must be equal to the arc-length of some arc contained between T. and the point indicated by 𝔛.) The triangle I.H.n. is similar to the triangle A.B.n. and the bases of the two triangles together—to wit H.n. and B.n. together—include all of the magnitude contained in the line R.T., and, since 𝔛. is the divisional point which divides the part of the line R.T. belonging to R, from the part thereof belonging to T, the ratio of R.𝔛. to T.𝔛. must be the same as the ratio of B.n. to H.n. and the same as the ratio of the radius A.B. to the radius I.H. Therefore the point 𝔛 has a similar relation to each of the arcs, and to the similar lines belonging to each of the arcs respectively—to wit, to the arc H.e. and the arc B.S. . . . to the sine G.e. and the sine R.S. . . . to the 'diff. t.*x*. of the sine and arc-length' of the arc H.e., and the 'diff. S.*x*. of the sine and arc-length' of the arc B.S. . . to the 'diff. of the arc-length and tangent' *x*.m. and the 'diff. of the arc-length and tangent' *x*.T. Now, if the difference between the sine R.S. and the arc length of the arc B.S. were either less or greater than S.𝔛., then would *x*.m. be not in the same ratio to *x*.T. as the ratio of 𝔛 T. to 𝔛 R. (for if it be supposed possible that the tangent of the arc H.e. may be greater than T.m. then will t.m. no longer have the same ratio to T.t which S.T. has to R.S.; nor will the ratio of t.m to S.T. be the same as the ratio of the radius I.H. to the radius A.B.). Therefore, since the sine R.S. and the *diff.*S.𝔛., of the sine and the arc, together with thediff. 𝔛.T. of the arc and the tangent belonging to B.S., wholly compound the line R.T., and since T.t. is to t.*x*. and to *x*.m., respectively, in the same ratio as R.S. is to S.*x*., and *x*.T. respectively, the point *x*. at the extremity of the arc-length of each arc cannot be other than the same point 𝔛. which divides the line R.T. in such wise that the part R.𝔛. thereof has the same ratio to the part T.𝒳. which the radius A.B, has to the radius I.H. *Wherefore it is demonstrated*, &c., Q.E.D.

Quantitive and numerical illustrations to the construction. *Fig.* 13.

By the Construction :—

R.T. = A.B. = 10·00000
R.S. = $\sqrt{50}$ = 7·07107

R.T.–R.S. = S.T. = 2·92893

$$\text{S.t.} = \text{S.b.} = \frac{\text{A.B.}}{10} = 1{\cdot}00000$$

1·92893 = T.t.

By Demonstration :—

$$\text{S.}\mathcal{X}. = \frac{\text{R.S.}}{9} = \frac{7{\cdot}07107}{9} = 0{\cdot}785674$$

R.𝒳. = (R.S. + S.𝒳.) = (7·07107 + 0·785674) = 7·85674
Now *t.x.* : T.t. : : S.𝒳. : R.S.

$$\text{Therefore}\frac{1{\cdot}92893 \times {\cdot}785674}{7{\cdot}07107} = {\cdot}214325 = \text{t.}\,x.$$

T.*x.* = (T.t + t.*x.*) = (1·92893 + ·214325) = 2·14325
And T.𝒳. = (S.T.–S.𝒳,) = (2·92893–·785674) = 2·14325

Test : By adverse supposition :—

$$\text{Let it be possible for S.}\mathcal{X}.\text{ to be less than}\frac{\text{R.S.}}{9} = \frac{707107}{9}$$

and let S.𝒳. = 0·78283 Then
R.𝒳. = (R.S. + S.𝒳.) = (7·07107 + 0·78283) = 7·8539
Now t.*x.* : T.t. :: S.𝒳. : R.S.

$$\text{Therefore}\frac{1{\cdot}92893 \times {\cdot}78283}{7{\cdot}07107} = {\cdot}213549 = \text{t.}\mathcal{X}.$$

and T.𝒳. = (T.t. + t.𝒳.) = (1·92893 + ·213549) = 2·14248
But T.𝒳. = (S.T. –S.𝒳.) = (2·92893 – ·78283) = 2·14610

Therefore the same line T.𝒳. has two different quantities of magnitude, which is impossible. In the same manner it may be shown that $S.\mathcal{X}.$ cannot be greater than $\frac{R.S.}{9}$ .... Wherefore $S.\mathcal{X}. = \frac{R.S.}{9}$

Illustrations of the relation of the arc-length to the sine of the half-quadrant. By the construction, Fig. 13.

It is evident that, since *s.x.* : *S.x.* : : *s.b.* : *A.B.*, if another arc be described with the point *s.* as a centre, and a radius *s.c.* less than *s.b.* in the same ratio that *s.b.* is less than *A.B.*, then *s.x.* must contain the sine and arc-length of the arc so described with the radius *s.c.*; and, again, the diff. *f.x.*, between the sine and arc-length of this small arc must contain the sine and arc-length of a smaller arc, described with the point at the extremity of the sine of the last small arc as a centre, and with a radius less than *s.c.* in the same ratio that *s.c.* is less than *s.b.*; and, in the like manner, arcs may be continually described, each arc being less than the arc preceding it in the same ratio, so long as there be any assignable quantity of distance remaining between the extremity of the sine of the arc last described and the point *x.*

Note.—The reader may, if he please, describe Fig. 13, on a scale ten times larger; the small arc and radius, *b.s.* and *s.b.* will be then enlarged to the size of the greater arc *B.S.* and the greater radius *A.B.*, and the second smaller arc, described with radius *s.c.*, will be then, if described, the same size and occupy the same relative position in the enlarged figure, which the arc *b.s.*, described with the radius *s.b.*, occupies in our Figure 13. The figure may be then again enlarged in the same proportion as before and a third smaller arc be described; and so on 'ad infinitum.' Instead, however, of actually describing enlarged figures, a mental alteration in the letters denoting the parts will serve the purpose, by supposing the lesser arc to have been enlarged into the greater arc, as often as may be desired.—

Quantitive and Numerical Illustration.

$$(R.\mathscr{X}. + T.\mathscr{X}.) = R.T. = 10\cdot0000.$$

$$\text{Since } R.\mathscr{X}. \text{ equals } R.S. + \frac{R.S.}{10} + \frac{R.S.}{100} + \frac{R.S.}{1000} + \text{ ad inf.}$$

$$\text{And } T.\mathscr{X}. \text{ equals } T.t. + \frac{T.t.}{10} + \frac{T.t.}{100} + \frac{T.t.}{1000} + \text{ ad inf.}$$

$$\text{Therefore}\left\{\begin{array}{l} 707107 \\ \quad 707107 \\ \qquad 707107 \\ \qquad\quad 707107 \\ \qquad\qquad 707107 \\ \qquad\qquad\quad 707107 \\ \qquad \&c.,\&c., \\ \text{`ad infinitum.'} \end{array}\right\} + \left\{\begin{array}{l} 192893 \\ \quad 192893 \\ \qquad 192893 \\ \qquad\quad 192893 \\ \qquad\qquad 192893 \\ \qquad\qquad\quad 192893 \\ \qquad \&c.,\&c., \\ \text{`ad infinitum.'} \end{array}\right\} = 10\cdot0000$$

(Fig. 14.)—CONSTRUCTION.

With centre *A.* and radius *A.B.* describe the quadrant *B.M.F.* . Bisect the quadrant in the point *M.*, and through *M.* draw *A.K.*, the secant. Draw, also *M.N.*, the sine of the arc *B.M.*, intercepting *A.B.* at *N.* Divide *A.N.* into ten equal parts, and divide also *N.B.* into ten equal parts, at the points of equal division, 1.2.3. 4.5.6.7.8.9., respectively:—

With radius 9.9. describe a quadrant of one-tenth less magnitude than *B.M.F.*

With radius 8.8. describe a quadrant of two-tenths less magnitude than *B.M.F.*

With radius 7.7. describe a quadrant of three-tenths less magnitude than *B.M.F.*

And so on. Finally :—

With radius 1.1. describe a quadrant of nine-tenths less magnitude than *B.M.F.*

To each successive arc draw the tangent and cotangent of the arc, and also from each of the points in the line *M.S.*, where the line is intersected by the successive arcs, draw a perpendicular through the line *B.K.*

We now obtain three distinct divisions of *B.K.*;

(1) the successive tangent-lengths dividing *B.K.* into ten equal parts;

(2) the successive arc-lengths dividing *B.O.* into ten eqaal parts;

(3) the successive sine-lengths dividing *B.D.* into ten equal parts.

Taking the radius *A.B.* = 10. The quantitive relations of the three lines are *B.K.* = 10·000000
B.O. = 7·856742 *B.D.* = 7·071068

And these relative proportions are precisely the same in the least as in the greatest of the arcs, and however far the reduction in size may be carried, so long as it be possible to describe a similar arc, or to imagine a similar arc to be described, these lines pertaining to the arc, and which even in imagination are inseparable therefrom, must necessarily have the same relative proportions each to each and each to the arc. Referring to Fig. 10, we observe that so soon as the quadrant *B.M.F.* commences to roll upon the line *B.E.*, the upper half-quadrant *M.F.* commences at *M.* to pass through the line *M.S.*; when the rolling process is completed, and the point *M.* becomes in contact on *B.E.*, the entire arc has passed through the line *M.S.*, and every component part of the arc has passed through the line at the same angle; therefore if we suppose a number of perpendiculars drawn through the line *M.S.* at any very minute distances from each other, each proportionally minute part of the arc in passing through the line *M.S.* would form with that line and the perpendicular (which cuts the line where the minute portion of the arc commences to pass through,) the figure *R.O.S.* on a scale proportionate to the minute magnitude of the lines compounding it. Now when the point *M.* of the arc has arrived at *O.* the straight* longitudinal extension of the arc is contained in the length of the line—to wit, in (*O. T.*) *R.S.* Every portion of the arc, therefore, relatively to the straight longitudinal measurement thereof on the line *O.T.* suffers a diminution in length in the ratio of *R.S.* to *M.S.* Consequently a manifest relationship becomes apparent between the horizontal and perpendicular lines *M.S.* and *R.O.*, which must be so proportioned to each other as to result in the arc length *O.S.* containing as its sine the horizontal length *R.S.* The quantitive relation expressed in figures makes the magnitudinal relation clearly apparent.

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* We mean by this expression—the longitudinal space between two perpendiculars, the one drawn through the point at one extremity of the arc, and the other (perpendicular) drawn through the opposite extremity of the arc.

for let *A.B.*, the radius, = 10, then :—
M.S. the arc length = 7·85674
R.S., the sine. = 7·07107
R.O. the versed sine = 2·92893
Now *R.S.* + *R.O.* = *A.B.*
thus, 7·071068 + 2·92893 = 10·00000

$$(\cdot 7856742 - \frac{\cdot 7856742}{10}) \times 10 = 7\cdot 071068$$

That is. . nine-tenths of the diff. of the sine and arc (of the octant) multiplied by ten, equals the sine.

The quantitive relation of the other principal lines which occur in these figures (Figs. 10 and 14) may be briefly noticed. The secant *A.K.* is equal to the chord of the quadrant which relatively to the radius [*] as 10, contains $\sqrt{200} = 14.14214$ (equal to twice the sine of the arc *B.M.*) and, therefore, if divided into nine equal parts, the quadrant contains ten equal parts, each equal to each of those nine parts (Coroll. prop. *D.*) Again, the difference of the radius *A.M.* and secant *A.K.*, namely *M.K.*, is in the same proportion to *M.D.* as *A.M.* : *AN.*, . .

---

[*] It may be observed that taking the radius = 1. The chord of the quadrant becomes $\sqrt{2}$, and the sine of the half-quadrant becomes $\sqrt{\frac{1}{2}}$.

to wit, as 10 : 707107. Therefore, $M.K. = 4.14214$. (*i. e.* $\sqrt{200} - 10$). $M.D. = 2.92893$. . . The numerical values of these quantities of magnitude in units of radius exhibit their relationship, thus: $R.S. + RO. = A.B.$(or $B.K.$)
$7{\cdot}07107 + 2{\cdot}92893 = 10$

$$\frac{M.\ K. \times M.\ N.}{B.\ K.} = M.\ D.$$

$$4{\cdot}14214 \times \frac{7{\cdot}07107}{10} = 2{\cdot}92893.$$

MATHEMATICS AND THE ART OF COMPUTATION.

The fundamental character of the relation of the circle to the science of number and quantity is established by demonstration that the difference of the quadrant and the chord of the quadrant, (of the arc and sine of the arc) is an aliquot part of the quadrant and of the chord, and that the number of those equal parts contained in the chord being nine—the quadrant contains ten; because herein we find conclusive evidence that the (so-called) Arabic system of notation is not an artificial human contrivance, but a great natural fact of a primary character, a fundamental part of the Science of Creation. When this is well understood it will be only necessary, in order to appreciate in some measure the immense number of facts of a secondary character, belonging to the science of Number and Quantity, furnished by the correlation of the lines compounding (or belonging to) the circle, to consider the relationship of the few primary lines—exhibited in their numerical values furnished above—in connection with the method shown in Fig. 14, of reducing the arc-magnitude *B.M.* from a radius equalling 10. to the one-tenth thereof by a gradation of similar arcs, through the nine successive intermediate magnitudes. It is to be noted that each of these lesser arcs has its own dependent lines with the same respective proportions each to each as the lines belonging to the greater arc, and that since each lesser arc has a known definite ratio to the greater arc, each and every line belonging to each lesser arc has a known* definite quantitive (numerical) ratio to each and every line belonging to the greater arc. Again, this primary division of the arc of given magnitude (that of the arc with radius = 10, into units) may be subjected to subdivision, and then, to further subdivision; each division furnishing an additional series of quantitive representatives or members, each member of the series having its own system of compounding lines, with their definite ratios each to each, and each to the radius of that member; and each member also having a definite known ratio * to each of the other members of that series, and, through the primary member of that series, having a definite known ratio to the general primary,—that is, to the primary arc or circle of given magnitude; and, through the general primary, having also, a known definite ratio to each of the members, and to each and all the definite divisions and subdivisions of the primary circle.

Now these relations of the compounding lines belonging to the circle are natural facts. They may be justly considered the *material* out of which and with which the instruments of quantitive analysis are to be constructed, to be improved, to be simplified, and to be, as far as possible, perfected.

That these facts are existent, and that they are not by any means deeply buried, is well known to many of those whose occupation it is to *till the ground*, or whose especial duty it is so to do; but many of those persons prefer, with a strange, and, as it would seem, with an increasing perversity, to cultivate the *thorns and thistles*, leaving the *good seed* as not worth utilizing.

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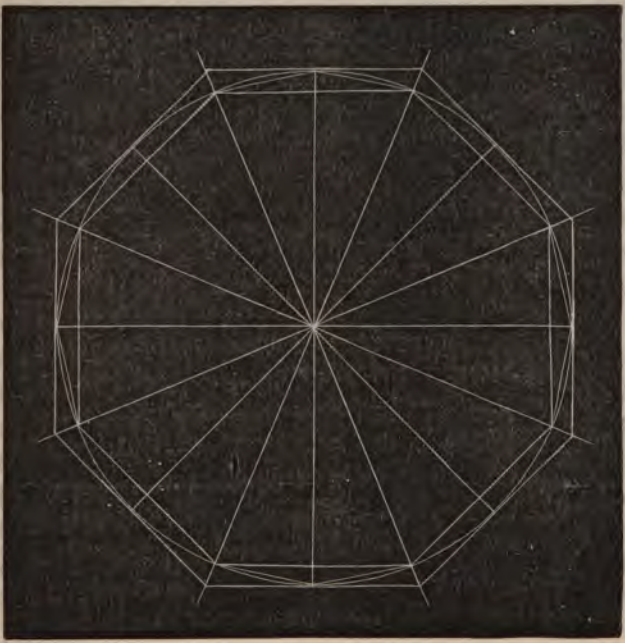
* Or which may be made known. (*In this connection. . see Cyclometry, (Part Fourth.)*

Amongst the things denominated 'thorns and thistles' we are not to be understood as intending to include the modern methods of mathematical analysis. It is true that in the compound trigonometrical process (under which general expression it is intended to include collectively all of the several forms of applied quantitive mathematics) we have an instrument of great value, a mechanism which is correctly adapted to perform its work with certainty and accuracy, and which is, when properly used, perfectly reliable and trustworthy. It is also true that this mechanism has performed a part in the work of civilization, the value and importance of which, in a human sense, it might be difficult to overstate or to estimate too highly; and it is again also true that the lives of very many able men belonging to successive generations have been employed in constructing, elaborating and perfecting this mechanism, and that it has been now brought, comparatively speaking, to a state of completeness and, almost, of perfection. But all this being admitted and fully appreciated,—it is no less true that this mechanism is only a human contrivance,—it is one form of application only out of a number in which the *material* may be applied; it is the utilization only of a very few facts where the number, from which selection may be made, is almost unlimited. That the mechanism is of a complicated and elaborate character requiring in its higher forms a special training, a long course of study, and much practice to apply it efficiently and with safety to the purpose for which it is adapted, is undeniable. Is this a recommendation or the reverse? No doubt its application and use had much better be left to those who have had the necessary training and are therefore able to apply it in the proper manner: but is it to the advantage of the community that such potentiality should be confined to a very few individuals? Is there any obvious natural necessity that it should be so? Who shall say that the few facts now utilized are the best adapted for the purpose of such an instrument, or that the form of mechanism now in use is the most simple and effective that can be devised. The evil of separating one division of knowledge from the rest as an exclusive department is manifold and unquestionable, and that it is so would probably be admitted in theory by some, perhaps be readily admitted by many, who are, nevertheless, quite prepared to say to those of their fellow labourers in the field of science who approach too nearly their department.—'This is holy ground—go back: thou art unfit.' Is it, or is it not, true that the language of mathematics is fast becoming an unknown tongue to ordinarily educated men, and that those to whom it is known can scarcely hold converse with their fellows (on any scientific subject) in ordinary language without a feeling of condescension and scarcely without a feeling of impropriety? No man's knowledge is perfect, or nearly perfect, in respect even to one kind or variety of knowledge, and much of the value of any one kind unquestionably consists in its belonging to general science, as one member of that body to which all the members belong. Is it true that the mathematician does now, *in some degree*, regard his fellow-worker who is unpracticed in the calculus and non-conversant with differential methods as but little better than a publican and heathen? If it be true that such feeling does manifest itself in any considerable degree, it may be pronounced decidedly unwholesome and bad—bad for science and bad for civilization—because mathematical knowledge is a necessity to science and a necessity to civilization.

We have now put before the public, demonstration of the problem which determines the quantitive ratio of the perimeter of the circle to the diameter. We know that examination will show the demonstration to be mathematically incontestable, and we know, moreover, that the requisite examination cannot be now much longer delayed. We are mindful, however, that there has been for a long time past a trigonometrical (supposed) solution of the problem, which is looked upon by many persons, educated in this department of science, as a mathematical determination of the question; so that when those persons are asked to examine and consider a geometrical solution of the problem, they reply by requiring to have that which they consider to be adverse demonstration disproved before they are willing to pay any attention at all to the alleged geometrical demonstration. In the circumstances of this particular case there is, at least to some extent, a justification of such a requisition as reasonable, because mathematicians have been exposed to much and continual annoyance on this subject from persons who, having neglected the legitimate means of qualifying themselves to investigate such a subject, have not hesitated to give trouble to and waste the time of other people, without first taking the trouble themselves to consider whether their qualifications were such as to entitle them to occupy the attention of others, or to justify them in endeavouring to do so. We will now, therefore, attend to this requisition.

There are several methods in which the trigonometrical process is applied, from each and all of which the same result (supposed solution) is obtained. But these several methods belong essentially to one and the same process, of which they are merely variations. That process consists in dividing and subdividing the arc of a circle, and in measuring by trigonometry the extremely small portion of the arc, which would result from a long continued process of reduction by bi-sectionl division. The application of the process being fundamentally based on an assumption or assumed axiom—that, if the arc of a circle be continually reduced by continued bi-section, the last remaining fraction of the arc is equivalent to the fraction of a straight line. The truth of this fundamental assumption we deny. In regard to the trigonometrical result, our objection is to the application of that result, and to the inference that a correct solution of the question is thereby arrived at. The most satisfactory way to specify our objection and to point out the precise locality of the mistake, will be to take Legendre's method of applying the process by inscribing and circumscribing polygons in and about the circle; because Legendre's formal exhibition of the process so applied is generally considered by mathematicians as particularly conclusive and reliable.



LEGENDRE'S GEOMETRY, (translated by D. Brewster.)

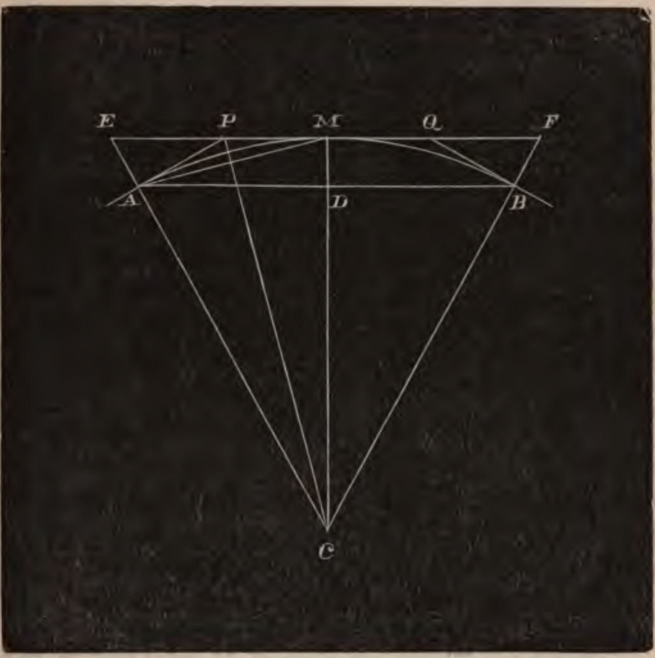
BOOK V.—PROP. XIII. PROBLEM.—

"*The surface of a regular inscribed polygon and that of a similar polygon circumscribed, being given; to find the surfaces of the regular inscribed and circumscribed polygons having double the number of sides.*

Let *A.B.* be a side of the given inscribed polygon; *E.F.*, parallel to *A.B.*, a side of the circumscribed polygon; *C.*, the centre of the circle. If the chord *A.M.*, and the tangents *A.P.*, *B.Q.*, be drawn, *A.M.* will be a side of the inscribed polygon, having twice the number of sides; and *A.P.* + *P.M.* = 2 *P.M.* or *P.Q.* will be a side of the similar circumscribed polygon, (Prop. VI. Cor. 3.) Now, as the same construction will take place at each of the angles equal to *A.C.M.*, it will be sufficient to consider *A.C.M.* by itself, the triangles connected with it being evidently to each other as the whole polygons of which they form part. Let *A.*, then, be the surface of the inscribed polygon whose side is *A.B.*, *B.* that of the similar circumscribed polygon; *A.′* the surface of the polygon whose side is *A.M.*, *B′.* that of the similar circumscribed polygon: *A.* and *B.* are given, we have to find A′. and *B′.*

First: The triangles *A.C.D.*,*A.C.M.*, having the common vertex *A.*, are to each other as their bases *C.D.*,*C.M.*, they are likewise to each other as the polygons *A.* and *A′.* of which they form part: hence *A* : *A.′*: : *C.D.*: *C.M.* Again, the triangles *C.A.M.*, *C.M.E.*, having the common vertex *M.*, are to each other as their bases *C.A.*, *C.E.*; they are likewise to each other as the polygons *A′.* and *B.* of which they form part; hence *A′* : *B.* :: *CA* : *C.E.* But since *A.D.* and *M.E.* are parallel, we have *C.D*:"

“ $C.M : : C.A. : C.E$,—hence $A : A' : : A' : B$—hence the polygon $A'.$, one of those required, is a mean proportional between the two given polygons $A.$ and $B.$, and consequently $A'. = \sqrt{A. \times B.}$



Secondly.—The altitude $C.M.$ being common, the triangle $C.P.M.$ is to the triangle $C.P.E.$ as $P.M.$ is to $P.E.$; but since $C.P.$ bisects the angle $M.C.E.$ we have $P.M. : P.E. : : C.M. : C.E.$ (Book IV., Prop. XVII.) $: : C.D. : C.A. : : A. : A'.$—hence $C.P.M. : C.P.E. : : A. : A'.$—and consequently $C.P.M. : C.P.M. + C.P.E.$ or $C.M.E. : : A. : (A. + A'.)$ But $C.M.P.A.$, or $2\ C.M.P.$ and $C.M.E.$ are to each other as the polygons $B'.$ and $B.$ of which they form part; hence $B' : B. : : 2\,A. : A. + A'.$ Now $A'.$ has been already determined; this new proportion will serve for” "determining B'. and give us $B'. = \frac{2\ A.B.}{A. + A.}$ and thus by means of the polygons A. and B. it is easy to find the polygons A'. and B'. which shall have double the number of sides.

Prop. xiv. Problem.

To find the approximate ratio of the circumference to the diameter.

Let the radius of the circle be 1; the side of the inscribed square will be $\sqrt{2}$ (Prop. III. Sch.,) that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let us therefore put $A. = 2$ and $B. = 4$; by the last proposition we shall find the inscribed octagon $A' = \sqrt{8} = 2{\cdot}8284271$, and the circumscribed octagon $B'. = \frac{16}{2 + \sqrt{8}} = 3{\cdot}3137085$. The inscribed and the circumscribed octagons being thus determined, we shall easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put $A. = 2{\cdot}8284271$. $B. = 3{\cdot}3137085$; we shall find $A' = \sqrt{A.B.} = 3.0614674$, and $B'. = \frac{2\ A.B.}{A. + A'.}$ $= 3{\cdot}1825979$. These polygons of 16 sides will in their turn enable us to find the polygons of 32—and the process may be continued till there remains no longer any difference between the inscribed and the circumscribed polygon, at least so far as that place of decimals where the computation stops, and so far as the seventh place, in this example. Being arrived at this point, we shall infer that the last result expresses the area of the circle, which, since it must always lie between the inscribed and circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place."

"We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals:

Numbr of sides.	*Inscribed polygon.*	*Circumscribed polygon.*
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

The area of the circle, we infer, therefore, is equal to 3·1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the area of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3·1415926, when the radius is 1; or the whole circumference must be 3·1415926, when the diameter is 1; hence the ratio of the circumference to the diameter, formally expressed by π, is equal to 3·1415925. The number 3.1416 is the one generally used."

Of these two propositions the last is a computation* based upon the first, or it may be considered as the numerical equivalent and illustration of the first, exhibiting in numerical units of the radius those relations of the parts which the first proposition apparently establishes. The first of these two propositions (prop. XIII.) is, however, quantitive, and may be also correctly considered as belonging to the science of Quantity and Number† rather than to that of 'Magnitude and Form.' Our objection to these propositions is two-fold, by which we mean that we object to two assumptions involved in these propositions, and which two assumptions, although nearly related, may be considered distinct.

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* All computations really such (*i.e.* which are not merely explanatory or merely the numerical equivalents of state-

Almost at the commencement of proposition XIII., we find the first assumption, to which we object, distinctly stated in these words: "If the chord *A.M.* and the tangents *A.P.*, *B.Q.* be drawn, *A.M.* will be a side of the inscribed polygon, having twice the number of sides; and $A.P. + P.M. = 2\ P.\ M.$ or *P. Q.* will be a side of the similar circumscribed polygon." Here then we have the statement that $A.P. + P.M. = P.\ Q.$ To justify the acceptance of this statement it should either be supported by demonstration or be in itself manifestly true. (1) Is it supported by demonstration? We refer as directed to Cor. 3 of Prop. VI., which reads thus: "It is plain that $N.\ H. + H.\ T. = H.\ T. + T.\ G. = H.\ G.$ one of the equal sides of the polygon," &c. Now these

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ments), may be considered the solutions of propositions, in the science of quantity and number. If an algebraical computation the proposition is quantitive; if belonging to arithmetic, the proposition is numerical.

†It is not meant that these propositions of Legendre are, for this reason, less reliable or of less value, but the distinction is noted as a protest against calling things which belong to different divisions of science, and which are not the same, by the same name. According to the title of the book, they purport to be propositions in Geometry.

lines are in the same case as *A. P.* + *P. M.* = *P. Q.*; so that we find therein not demonstration, but an assertion that the thing stated is a manifest fact. (2) Is it manifestly true? The case in the Corollary, just quoted from, is that *N. H.* + *H. T.*, which is an angle and indirectly defined, by the preceding and following propositions, to be a complete angular figure * is equivalent to *H. G.* which is a straight line. Similarly in proposition XIII., the angle *A. P.* + *P. M.* is assumed to equal the straight line *P. Q.*, because either *A. P.* or *P. M.* measured separately as a straight line is apparently equal to one-half the straight line *P. Q.* It is, therefore, assumed as manifest that the vertex of the angle has no quantitive value in itself independently of its sides considered as straight lines.

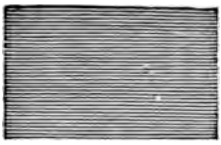

Fig. 1.

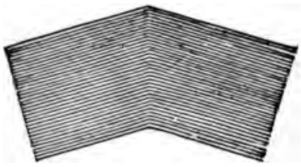

Fig. 2.

Let us suppose Fig. 1 to be a figure compounded of straight lines placed close together, and Fig. 2 to be a figure likewise compounded of angle-lines placed close together. In Fig. 1, if two of the lines be considered relatively to each other, they appear to be similar and equal magnitudes, and on consideration of the figure,—that is, of all the lines compounding the figure relatively to each other,—it becomes manifest that the lines are actually similar and equal longitudinal magnitudes, because both the sides of the figure compounded by the lines are perpendicular to the longitudinal extension of those straight lines, and the compounded figure is rectangular:

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*That is—not a compound fragmentary figure formed by two lines merely placed together.

therefore, if two or more of the lines be similarly divided, the divisional part or parts of the one will be equal to the similar divisional part or parts of the others. In Fig. 2, if the two top or bottom angle-lines be considered relatively to each other they appear to be also similar and equal magnitudes, but on consideration of the figure, that is, of all the lines compounding the figure relatively to each other, it becomes manifest that the lines, although similar, are not equal, for the lower lines of the figure are evidently lesser longitudinal magnitudes than the upper lines; nevertheless, if two of the angles compared together be considered as compounded of four straight lines, and the four straight lines be compared with each other as longitudinal magnitudes, the difference may be less than any assignable quantity; moreover, if the equal lengths of the lines be equally increased, the extremely minute difference must be proportionately diminished, and if the equal longitudinal magnitudes be in such wise indefinitely increased, the difference will be indefinitely diminished; nevertheless, the diff. is actual, and if one of the angle lines compounding the figure, however extremely minute may be the breadth which that line is imagined to possess, be (supposed) separated into two lines each having half that breadth, then *must necessarily* the outer of the two halves be of greater length than the inner; and, moreover, the under surface of the outer half must necessarily be in longitudinal magnitude greater than the upper surface of the inner half. Before considering the (1) objection further, we will go on to define the character of the (2) objection which, as we have stated, is in some measure distinct, and we will then consider the two-fold objection as one. In Fig. 3, we have the arc *A. B.*, the sine *C. B.* and the tangent *A. d.* of the arc. Now, there is an assumption, which we are about to explain and to which we are about to object, in respect to the diminution of this figure by definite division of the arc, which is at the present time generally adopted by mathematicians; and which, although not distinctly put forward as an axion or theorem in either of the two propositions, is adopted also by Legendre and forms an essential part of the foundation* upon which his apparent demonstration rests. Let us suppose the arc *A. B.* to be bisected, and the sine of the remaining half-arc to be drawn, and the tangent *A. d.* to be also bisected; we shall then find that both the sine and tangent in the lesser resulting figure are much nearer to the arc throughout their length than in Fig. 3. If the arc and the tangent of the lesser figure be also bisected, a still nearer approach of the sine, the arc, and the tangent to each other, will result, and it

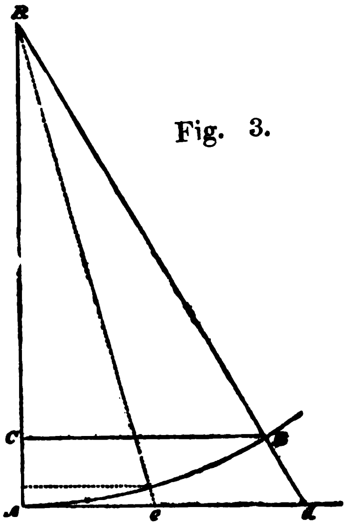

Fig. 3.

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* The coincidence of the lines appears to be included in and to be partly the subject of his demonstration; but careful consideration will show that the process itself, having regard to its intended application, is primarily based upon a foregone conclusion as to such coincidence.

is evident that if this process of bisection were to be continued in like manner, the deviation of either of the remaining parts of the three lines from a single straight line would be but very small; for, at the 8th bisection, only the 256th part of the tangent *A. d.* would remain, and this would evidently very nearly coincide with the remaining 256th part of the arc. Now the assumption which we wish to specify, and to which we object, is that—if the process of bisection be so continued, a small part of each of the three lines will eventually remain, which will absolutely coincide and become essentially one line. Observation of Fig. 3 shows that the two extremities of the figure *A.C.B. d.* are not similar; the relations of the three lines—the sine, the arc, and the tangent—are such that if the line *C. B.* be moved down and applied upon the line *A. d.*, a small part of the three lines at the extremity *A.* will appear to coincide; but at the opposite extremity *B.* although the two lines *C.B.* and *A.d.* would in that case coincide, the arc *A. B.* would deviate by the amount of its curvature from the straight line. Now if the usual method of considering the result of the process is reversed, and, instead of considering the eventual relationship of the lines (when the process has been carried very nearly to the vanishing point) with reference to the extremity *A.* and the perpendicular *R. A.*, that eventual relationship is considered with reference to the opposite extremity of the remaining arc and to the radius which intercepts that opposite extremity—the impossibility of an absolute coincidence of the three lines becomes at once apparent, because—however minute the fraction of the arc remaining—*so long as* there be *any arc*, the radius which intercepts the extremity of the arc opposite from *A.* can never become quite perpendicular, consequently the sine must necessarily remain, until the very last, inside the arc, and the tangent must remain outside. The obvious impossibility of an actual coincidence of the three lines would seem to have forced itself for the moment (to be again immediately lost sight of) on the attention of Legendre, and it is admitted in Prop. XIV. by the words " We shall infer that the last result expresses the area of the circle, which, *since it must always lie between the inscribed and circumscribed polygon,* *and since these polygons agree as far as a certain place of decimals, must also agree with both as far as the same place." Now the cumulative character of any disagreement has here been overlooked; a very close approximation to equality in the three lines, left remaining as the result of continued bisection carried to the extreme, is quite intelligible and indisputable, but a very close approximation between the inscribed polygon, the circumscribed polygon, and the arc between them, if that arc contains the eighth part of a circle, becomes, when the construction of the circle is correctly understood, a manifest impossibility. If the increase of the small remaining part of the arc was merely an extension thereof into a greater magnitude of similar form, such as would result from increasing the length of the radius, namely, the extension into a larger arc similar in form to the small arc, or more strictly speaking, into the similar fraction of a larger circle, there would then be a possibility of an almost absolute agreement, such as alleged; but a circle cannot be produced in such a way; the circle may be considered as formed by the longitudinal union of extremely small arc magnitudes similar in form to each other; consequently, as the compounded magnitude increases, so does the deviation from the straight line, and so, also, does the amount of difference between the arc and straight line, continually accumulate. The difference which expresses the disagreement in actual length of the small lines, is a part of the *relation in form* of those lines, which increases proportionally to the increasing development and magnitude of the figure. If the small part remaining as the result of bisection be the 100th part of the half-quadrant* then the difference between the sum of the one hundred small sines, the sum of the one hundred small tangents, and the half quadrant itself, will be one hundred times greater than the corresponding difference existent in the small arc between the similar lines; or if the small arc be the one-thousandth or the one-millionth part of the half-quadrant, then must the difference between the sum of the sines and of the tangents of all the combined small arcs, and the half-quadrant itself be one thousand times, and one million times respectively, greater than the difference between the corresponding lines, to wit—between the sine, the tangent, and the arc-length, belonging to the small arc.

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*We have put these words in italics to specify the admission. It is to be noted that this admission is quite irreconcilable with the alleged agreement or equality of the lines, because such agreement would be equivalent to absolute coincidence of the lines.

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* The circle naturally divides itself into eight parts compared to an inscribed or circumscribed square, because the four quadrants are each divisible into two half-quadrants, each of which has the same relative relation to the side of the square as the relation of the other half; so that the eight parts of the circle relatively to the sides of the square are similar and equal each to each, but if further division be made, then such equality and similarity in form relatively to the side of the square will no longer obtain, because the curvature of the diminished arc will be less than that of the part divided off from it. The non-appreciation of this fact has much to do with the erroneous conclusion supposed to have been demonstrated; it is, indeed, through the distinct and thorough appreciation of this relationship that an apprehension of the precise character of the fallacy is to be arrived at.

Some of those persons who are reasonable enough to feel that to nominally define a thing by calling it an abstraction does not define it at all, prefer to adopt the expression 'surface' whereby to denote a line, * and define a geometrical line as 'the surface of a geometrical figure.' Now the surface of a thing must either belong to and be a part of the thing or must be outside of it . . . in regard to the circle, for instance, the surface must either belong to and be a part of the area of the circle, or it must be outside the circle. If supposed to be within the circle then must the circumscribed polygon, compounded by the union of the small tangents, (if the circumscribed

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* Strictly speaking, according to the presently accepted doctrine, a line is the extremity of a superficies. If a superficies were to be defined as a real surface, compounded of real lines, the expression would not, we opine, be subject to objection ; a line might be then considered as one of the extremities of the surface, or as a section of, or as one of the elementary constituents of the surface. As the (so called)definition now stands, if we attempt to directly cognize it as an intelligible idea, we find ourselves almost immediately enveloped in a network of contradictions. (1) The surface is either compounded of lines, or it is not compounded of them.—If it is, how can that which hath breadth be compounded of that which is without breadth ? If it is not, what then is that surface of which the line is the extremity, and of which, nevertheless, the line is not a part ? of what then is that surface compounded, and is the surface itself a part of, or does it belong to anything ? But the difficulty (dilemma) is of a still more refined character.—" The extremities of a line are points," which are negatively *defined* to be *nothing*. Now, the relation of the line to the surface is defined to be similar to the relation of the point to the line : it therefore seems to follow that since the line has nothing for its extremities, the surface likewise has nothing for its extremities. How are we to cognize the idea of a surface without any extremities ? or, of a surface which hath length and breadth together with extremities which according to the definition, certainly have no breadth and probably have no length, for the supposition of a line without extremities possessing length is, if it have any meaning, the negative supposition of a non-existent line which might be possessed of length if it were existent.

(See, in the Appendix, Dr. Simson's explanation of a superficies and illustration of Euclid's dogma.)

polygon be a continuous and not a fractional figure) be necessarily greater than the surface. And if the surface be outside the circle, then must the surface be greater than the sine, which is always within the circle and therefore less than the circumscribing polygon by which the circle is surrounded.

The objections to Legendre's propositions, as a supposed demonstration of the ratio of the circle to the diameter, are :—

First.—Evidently, he assumes the circumscribed polygon to be a simple continuous figure ; because the application of his propositions is based upon a comparison of the polygon with the arc of the circle which is a simple continuous figure, and if the polygon be not simple and continuous his demonstration fails, since, in that case, the reasonableness of the application is not shown. It is objected that the case is not in fact as he assumes it to be : the circumscribed polygon is not simple and continuous, but compound and fragmentary . . for, if a straight line be bisected and the two halves thereof be placed together, so that their adjoining extremities form the vertex of an angle, it is manifest that the quantity of length contained in the exterior surface of that angle must be, measured as a continuous surface, greater than the sum of the two lines, measured separately and taken together ; because the interior surface (within the angle) must be equal to the lengths of the two lines taken together, and the exterior surface of an angle, considered as a complete figure, is evidently greater than the interior surface.

Second.—The inscribed polygon is assumed, if the process of bisection be continued, to become eventually coincident with the circle. It is objected that this is impossible, for so long as any part of the circle remains, however minute that part may be, the sine of the arc must be within the arc; and, if this be admitted, (which it must necessarily be) it then follows that, in comparing the half quadrant with the radius, the difference, because cumulative, must be increased proportionally to the greater curvature of the half-quadrant, compared with the lesser curvature of the minute arc.

Some mathematicians are disposed to require in this case, in addition to adverse demonstration and to reasonable objection, the explanation of an apparent difficulty of a particular kind. It is said :—there can be no doubt that this numerical quantity found by Legendre and others as the ratio of the circle to the diameter, does represent the ratio of a quantity which has some actual and significant relation to the circle; that it must be so, is established by indirect evidence, for the very same quantity appears as the notable result of quite a number of different computations connected with the circle. Explain, therefore, what this quantity really is. What is the relationship of this quantity if it be not in fact that assigned to it by Legendre and others?

The answer to this requisition is :—The quantity in question, namely—·78539 .... is the sum of the lengths of the sines of all the elementary arcs contained in the half-quadrant, and into which the half-quadrant may be divided by the continued process of bisection. In other words, ·78539 . . . . represents the length of the sine belonging to the extremely minute (ultimate) arc, which results from the continued process of bisection, multiplied by the number of those minute arcs contained in the half-quadrant, when the radius of the circle is valued as a unit.*

The origin of the error in Legendre's method, as well as in the application of the process known as that of continued bisection, is in the omission to observe that comparison has to be made between a continuous curved line (the circle) and a continuous straight line (the diameter.) The circle may be divided into four equal parts, and the inscribed (or circumscribed) square is then equally and similarly related to each of those parts; and, further, each of the quadrants and each side of the square may be bisected, and still the same equal and similar relation will exist between each of the eight parts of the curvilinear figure and each half-side of the square; but, if division be carried further, the relation will no longer continue similar and equal. If the half side of the square were to be broken up into its (ultimate) component parts, and these arranged as an inscribed (or circumscribed) polygon, each minute part would be related to its arc —similarly to each of the other minute parts; but the polygon, being fragmentary and non-continuous, would not be in the same case as, and would not admit of indiscriminate comparison with, a continuous straight line.

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* This fact, with its precise significance, may become more distinctly apparent by consideration of the process of the radial duplication of the bisected arc—an explanation of which will be found in the Appendix.

For the purpose of defining the characteristic distinction between a straight line and the arc of a circle, and, at the same time, indicating the boundary between the conceivable and the inconceivable—between the reality of Science and unreality of Metaphysics,—we will notice here the word 'infinite.' In the first place it may be of service to call attention to the foolish and mischievous manner in which this word is becoming more and more frequently used. The word 'infinite' is properly and correctly used in Mathematics and in other divisions of science as the opposite to 'finite.' Outside or independently of its use in such sense the word 'infinite' has a naturally sacred character as applying to the attributes of the Creator, and it is very desirable that the use of the word should be restricted to its proper and correct application. It is now used in all sorts of literary composition,—in newspaper articles, lectures, sermons, &c.—in a manner that may be termed *unsense*, and which would oftentimes be absurd if it were not calculated to have a seriously mischievous effect. Perhaps, the most common mistake is its use as equivalent to the expressions—'indefinite,' which means that which is finite, but of which the limit is not, or cannot be defined—and 'immeasurable,' which means that which cannot be measured in consequence of its exceeding greatness.

'Infinite' means boundless, unlimited, endless, continual extension, &c., &c. The word does not correctly compound with words (adjectives) which express comparative extent, or in which measurement is implied. Such a compound expression as 'infinitely great' is, therefore, (with exception of the theological sense) barbarous; 'infinitely greater' or 'wiser' is about equally so; 'infinitely small' and 'infinitely less,' or 'infinitely more foolish are perhaps even worse. We make these remarks in this place, primarily, for the purpose of obtaining the requisite attention to the distinction between a circle described (1) with a definite radius or (2) with a radius of indefinitely great length, and (3) a circle described with (using the expression for illustration only) an infinite radius. In the first (1) there is included, because necessarily resulting therefrom, the idea of a definite circle, of which the magnitude is determined and defined by the definite magnitude of the radius. Belonging to the second (2.) is a circle the magnitude of which is limited; its magnitude may be immensely, perhaps immeasurably, great, and therefore of unknown greatness, but it is limited by the limited length of the radius, whatever that length may be: for example, the radius, or radial distance, may be the distance between the earth and the most distant star, and, as the distance of the star is evidently finite, so must the magnitude of the circle described with that radial distance be limited accordingly. It may be here noted that we can conceive or adopt as a conception and distinctly cognize as an idea, anything of which we have (obtained) certain knowledge as a fact . . . taking as an example the case just stated, or its equivalent—of the earth moving in its orbit of revolution around the sun, if we consider the motion of the earth through a quantity (an extension) of space equivalent to a few yards or feet of terrestrial measurement, the motion would appear to be in a straight line. Compared to any merely terrestrial motion, it would be (almost absolutely) in a straight line; and, considered merely in reference to any such motions, its deviation from a straight line would be inconceivably small, and would scarcely admit even of intelligible expression as a comparison with any merely terrestrial quantity; nevertheless, by a knowledge of the fact, and of certain other facts with which to make comparison, we are enabled not only to obtain the conception of, but also to distinctly appreciate the deviation and, even, to determine and measure its amount with correctness and accuracy*; we are enabled to be perfectly sure that the path or line of the earth's motion, throughout that quantity of space, is not in a straight line but in the arc of a circle, having the sine of the arc within and the tangent of the arc without, the boundary (perimeter) of that circle.

Thirdly, (3) we have the idea of a straight line, or, so to speak, we have the compound idea† of the arc of a circle which has become a straight line. If it were conceivable that a radius-vector might have infinite length, it would then follow that the extremity of such a radius would describe a straight line, and the straight line might then be considered as the arc of a circle of infinite magnitude. It may be that herein we have the nearest approach that can be made by the human mind to a conception of that which, if it can exist, does not belong to the world of humanity; to a conception of the inconceivable; for, in what may be termed a compound negative sense, the mind does seem to be very nearly able to distinctly cognize the supposition as a positive idea. Having certain knowledge of the straight line as a fact, and apprehending the necessity that, if the supposition of the infinite radius were admissible, the straight line must have that relation to it which the arc of the circle has to a finite radius—the shadow seems to assume substance, and it is almost as though the mind were able to grasp the unreal idea as a reality. But, in fact, there would be no circle and the supposition as a positive hypothesis does not belong to (human) science. To entertain and to wilfully play with such an unreal idea as a positive or concrete conception is forbidden; to do so would be to leave the realm of science, and, contemning the guidance and authority of reason, to enter the dark domain of metaphysics. We believe, however, it is admissible to entertain the idea, thus far, negatively, for the express purpose of defining the boundary and of clearly realizing the essential distinction between the straight line and the circle.

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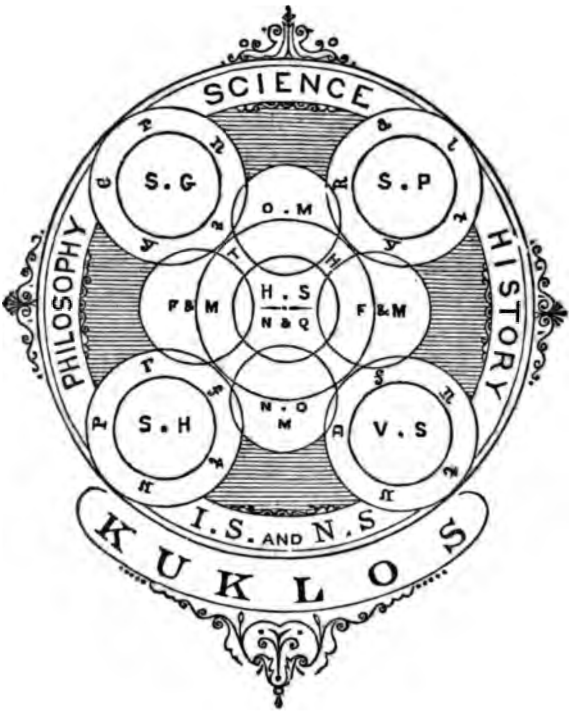
* Supposing, of course, the distance of the sun from the earth to be correctly and accurately known.

† We have already stated that these expressions are used, in this connection, only in illustration of the particular case under consideration.

A very serious obstruction in the way of intellectual progress has been now pointed out and clearly indicated. The false statement at the commencement of Euclid's great work is now unmasked and its actual character made manifest as a monstrous deception of Untruth in the place of that which it purports to be—a true definition of reality. Those who for the future are deceived by it must be so by wilfully subjecting themselves to such deception. So soon, however, as its false and deceptive character has become distinctly understood, those who more particularly represent Education and Science, should mark its baneful influence in the vantage ground it has so long occupied, and see to its speedy condemnation and removal. Let 'the stumbling-block be taken up out of the way.'



SUPPLEMENT

TO PART SECOND.

The Ultimate Tangent.—Objection to Legendre's mode of computing the *quantitive ratio of the polygon circumscribed about the circle, to the diameter of the circle.*

Now Legendre does not deny, but indeed distinctly admits, that the tangent must be always a greater longitudinal magnitude than the fractional arc which it subtends, however far the reduction of the arc by bi-sectional division may be carried; but he argues that when the remaining section of the arc becomes extremely small the curvature is almost eliminated, and the remainder of the tangent (or small side of the polygon) virtually (*i. e.* very nearly) coincides longitudinally with the remainder of the arc.

The fallacy in the application made by Legendre of the process adopted by him, may be said to consist in, or to arise from, the want of a complete apprehension as to the distinction between relative and absolute quantitive magnitude. All quantitve magnitude expressed as measurement is relative, and it is the expression of the ratio of one magnitude to another magnitude, which last is taken or given as the standard of comparison.

The object of Legendre's investigation is to ascertain the quantitive relation of the circumference of the circle to the diameter, expressed in units or standard increments of the diameter, or, which is equivalent, the relation of the octant to the radius expressed in units of radius. Legendre, increasing the number of sides of the circumscribed polygon by bi-sectional duplication, finally obtains a polygon of which one of the very small sides only very slightly exceeds in length the fractional arc which it subtends. He then measures the arc* and the tangent . . *i.e.*, he ascertains the comparative magnitudes of the small side and fractional arc, and the longitudinal value of the difference between them. How is that difference expressed? With what is the comparison made? Evidently with the diameter or radius of the circle circumscribed by the polygon; and is expressed in terms of that diameter or radius.

Legendre then, in order to obtain the quantitive relative magnitude of the complete polygon which circumscribes the circle, proceeds to multiply the quantitive contents of the one side by the number of the sides. Now the result of such proceeding is not what he supposes it to be, for the one side is connected, by magnitudinal relation, to the radius, in the units of which its quantitive value is expressed, and, consequently, by simply multiplying the side, he multiplies, together with it, the radius to which it is referred, and obtains, not a polygon of so many sides, but the one side of a polygon so many times greater . . *i.e.*, of a polygon which would circumscribe a circle of so many times greater magnitude. In other words; the one side of the polygon is *radially increased*, but it remains the same small fraction of the circle which it was before; and so if the multiplication were to be repeated an indefinite number of times, the relationship would remain the same; if the small fractional arc subtended by the one side in the first instance was the one-thousandth part of the circumference, so would the fractional arc subtended by the enlarged side be the one-thousandth part of the circumference, but the circle in the last case might be a thousand or a million times the magnitude of the first circle.

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* Strictly speaking he measures the sine and the tangent, and infers the comparative magnitude of the arc which is between them.

Legendre admits that the small fraction of the circle ..say the one-thousandth part thereof..contains a difference between the two magnitudes representing respectively the fractional arc and the one side of the polygon. Is it not evident that a complete circle circumscribed by a complete polygon must contain one thousand times that amount of difference? Not, be it observed, one thousand times in reference to a radius one thousand times greater, because then the proportion would remain the same..the entire compound magnitude would be increased collectively, all the parts alike, and the quantitive ratio of each to the other would be unaltered, whereas what Legendre intends to do is to build up a complete polygon from the one side by the addition to that one side of all the other sides without increase or alteration of the radius. Let us see in what manner this requires to be done. We will suppose the complete circumference of the circle to be six times greater than the radius, and the one small equal side of the polygon to be slightly greater than the six thousandth part of the circumference..therefore very nearly equal to the one-thousandth part of the radius. And we will suppose the difference between the small side (tangent) and the fractional arc which it subtends, to equal the six-millionth part of the small side (tangent). By multiplying the six-millionth by the six thousand we obtain the ratio of the difference contained by the complete circle between the circumference and the circumscribed polygon on the scale of the six-thousandth of the radius. That difference would be therefore equal to the one-thousandth part of the small side (tangent).* The scale has then to be increased to the magnitude of the original circle and its radius, which will be done by multiplying the quantitive values of the small circle and its circumscribed polygon, each by six thousand. The result will be the quantitive values of the (original) circle and of its circumscribed polygon..the difference between which will be equal to the one-thousandth of that polygon (which polygon will contain six thousand sides). But by Legendre's mode of computing, the result would be a difference of only the six-millionth part of the same polygon.

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* The small fractional arc is supposed to be divided into six thousand *similar* arcs. Now, to do this the original radius must be divided into six thousand parts. .then with one *of these divisional parts as a radius* the complete circle is described, and the small polygon is circumscribed about it by dividing the small side (tangent) into six thousand equal parts, and taking one of the divisional parts for each side of the polygon. It is

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evident that the former ratio of the difference..to wit, the six millionth part of the small side (tangent) relatively to the original radius..must be, relatively to the reduced radius, increased proportionally to the reduction of the radius: must be increased therefore six thousand times.

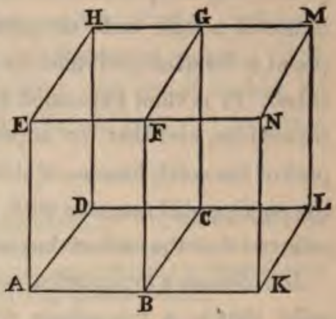
APPENDIX.

Euclid's definition of a superficies, a line, and a point.

An ingenious endeavour has been made by Dr. Simson (and others,) to give an intelligible explanation of the compound dogma prefixed to Euclid's work, by means of the figure of a solid cube or parallelopiped. The solid is supposed to be evenly divided by a section at right angles to its sides. It is then explained (argued) that the section is the superficies, and that the superficies cannot belong to either part of the solid, because if either one of the parts be removed the surface still remains with the other part; and hence it is inferred that the surface has no breadth; &c., &c., as follows:

Dr. Simson's explanation.—"It is necessary to consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, line, and superficies; for these all arise from a solid, and exist in it. The boundary, or boundaries, which contain a solid, are called superficies, or the boundary which is common to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies: thus, if BCGF. be one of the boundaries which contain the solid ABCDEFGH., or which is the common boundary of this solid and the solid BKLCFNMG., and is therefore in the one as well as the other solid, it is called a superficies, and has no thickness; for if it have any, this thickness must either be a part of the thickness of the solid AG., or the solid BM., or a part of the thickness of each of them. It cannot be a part of the thickness of the solid BM.; because, if this solid be removed from the solid AG., the superficies BCGF., the boundary of the solid AG., remains still the same as it was. Nor can it be a part of the thickness of the solid AG., because if this be removed from the solid BM., the superficies BCGL. the boundary of the solid BM., does nevertheless remain; therefore the superficies BCGF. has no thickness, but only length and breadth.

"The boundary of a superficies is called a line; or a line is the common boundary of two superficies that are contiguous, or it is that which divides one superficies into two contiguous parts: thus, if BC. be one of the boundaries which contain the superficies ABCD., or which is the common boundary of this superficies and of the superficies KBCL. which is contiguous to it, this boundary BC. is called a line, and has no breadth; for, if it have any, this must be part either of the breadth of the superficies ABCD. or of the superficies KBCL., or part of each of them. It is not part of the breadth of the superficies KBCL.; for, if this superficies be removed from the superficies ABCD., the line BC., which is the boundary of the superficies ABCD., remains the same as it was. Nor can the breadth that BC. is supposed to have be a part of the breadth of the superficies ABCD.; because, if this be removed from the superficies KBCL., the line BC., which is the boundary of the superficies KBCL., does nevertheless remain: Therefore the line BC. has no breadth; and, because the line BC. is a superficies, and that a superficies has no thickness, as was shewn; therefore a line has neither breadth nor thickness, but only length.



"The boundary of a line is called a point, or a point is a common boundary or extremity of two lines that are contiguous: Thus, if B. be the extremity of the line AB. or the common extremity of the two lines AB., KB., this extremity is called a point, and has no length; for if it have any, this length must either be part of the length of the line AB. or the line KB. It is not part of the length of KB.; for, if the line KB. be removed from AB. the point B. which is the extremity of the line AB., remains the same as it was; nor is it part of the length of the line AB.; for if AB. be removed from the line KB., the point B., which is the extremity of the line KB., does nevertheless remain: Therefore the point B. has no length; and, because a point is in a line and a line has neither breadth, nor thickness, therefore a point has no length, breadth nor thickness. And in this manner the definition of a point, line, and superficies are to be understood." *

We object to this explanation; that. .the boundary defined is a space outside the solid: for (1) the boundary which surrounds and contains a solid must be greater than and must be outside the solid which it contains, and (2) the boundary which separates and is common to the contiguous sides of two solids is a space—which space must contain more or less breadth according to the greater or less distance between the two solids (or two parts of the solid.) Let the distance between them be any definite small quantity of space......the two contiguous solids (or parts of the solid), may be then removed to twice that distance, and the boundary which is common to both will have twice the amount of breadth...... or, the two solids may be brought nearer together and the distance between them lessened to the one-half, the boundary common to both will then have one-half the breadth it previously had, or the distance may be lessened to the one-millionth part and then will the breadth of the boundary be diminished to the one-millionth part of the breadth contained in the previous boundary, and so on. But now, if the two solids (or the two parts of a solid) are brought into absolute

---

* But if the point which has no length is included in the line (which has no breadth or thickness) as a part of that line, it follows that the line itself has no length; therefore, it has no existence either as a natural or ideal reality—there is no line. It is of no avail to call it an abstraction, because the answer is plain—there is no abstraction. The compound dogma, so soon as reasonably examined, resolves itself into an unreal chimera of the imagination, *i.e.*, into nothing.

proximity and united into one solid (or into a whole undivided solid,) the boundary common to the two solids and which is outside each of the solids, and is a part of the boundary containing each of the solids respectively, has disappeared, it is no longer between them for they are united: evidently the boundary of the solid (or part) ABCDEFGH., if it be outside that solid must be now a part of the other solid (or part) BKLCFNMG., and inversely the boundary of the last, if it be outside the last solid itself, must be a part of the first.

The reasoning applies equally to the similarly related case of the *line*, in which the line BC. must be a breadth-containing space between the two contiguous superficies and be common to both of them; for if they be united, a boundary in the same sense is no longer existent; neither is it conceivable—for, to conceive *such* a boundary is to include the idea of separation which necessitates the cognition of a space measured by the distance (amount) of that separation. And it is evident that the space is the line so conceived or cognized.

Similarly in regard to the point B. If the point be the extremity of the line AB., and be in the line . . then is it a part of the line AB., and if the line AB. be removed, then is the point B. removed with it; for, to assert the contrary is to assert that the same one point B. can at the same time be in two or more different places, which is absurd. And again, if it be neither in the line AB., nor in the line KB., then it cannot be the extremity of either line, in the sense of belonging to and ending the line, therefore it must be the space between the extremities of the lines and must contain magnitude (breadth) limited and measured by the distance which separates those extremities, (or a minute and definite divisional part of that space may be taken to represent the line; i.e., the line may be considered as constituted by the space, or as contained in the space and constituted by a divisional part of the space.)

THE ARC-LENGTH AND ULTIMATE SINE.

The process of radially duplicating the bisected arc.—The result of this process is to obtain the sine length belonging to the minute (ultimate) part of the arc which would remain after the process of bisection had been repeated a great number of times—that is, repeated until the vanishing-point had been almost arrived at. It is, therefore, strictly speaking, *a method* belonging to the same general process of which several methods are already known and practised, amongst them being that of Legendre which we have quoted and examined.

The method which we are now about to explain has the advantage, we think, of exhibiting the facts from which the elements of the computation are derived, in a more simple, direct, and readily intelligible form; and perhaps, also, in a more generally instructive and useful form.

The method may be thus explained:—The half-quadrant (or any other definite fraction of a circle) being described with a given radius of definite length—(1) The chord of the arc is drawn. Through the terminal point of the arc, and at right angles to the chord, a line is drawn (of indefinite length,) which line is a part of the secant (or radius) belonging to half the arc radially duplicated in magnitude. (2) The arc is bisected and through the point of bisection, from the extremity of the perpendicular radius, a line is drawn which intercepts the secant of the duplicated half-arc; the line last drawn is therefore the chord of the 1st duplicated half-arc, and the point at which that line intercepts the secant is the terminal point of the 1st half-are of duplicated magnitude: that is, if the radius of the primary arc were to be duplicated * and a half-quadrant described therewith, the point thus found would necessarily bisect that half-quadrant.

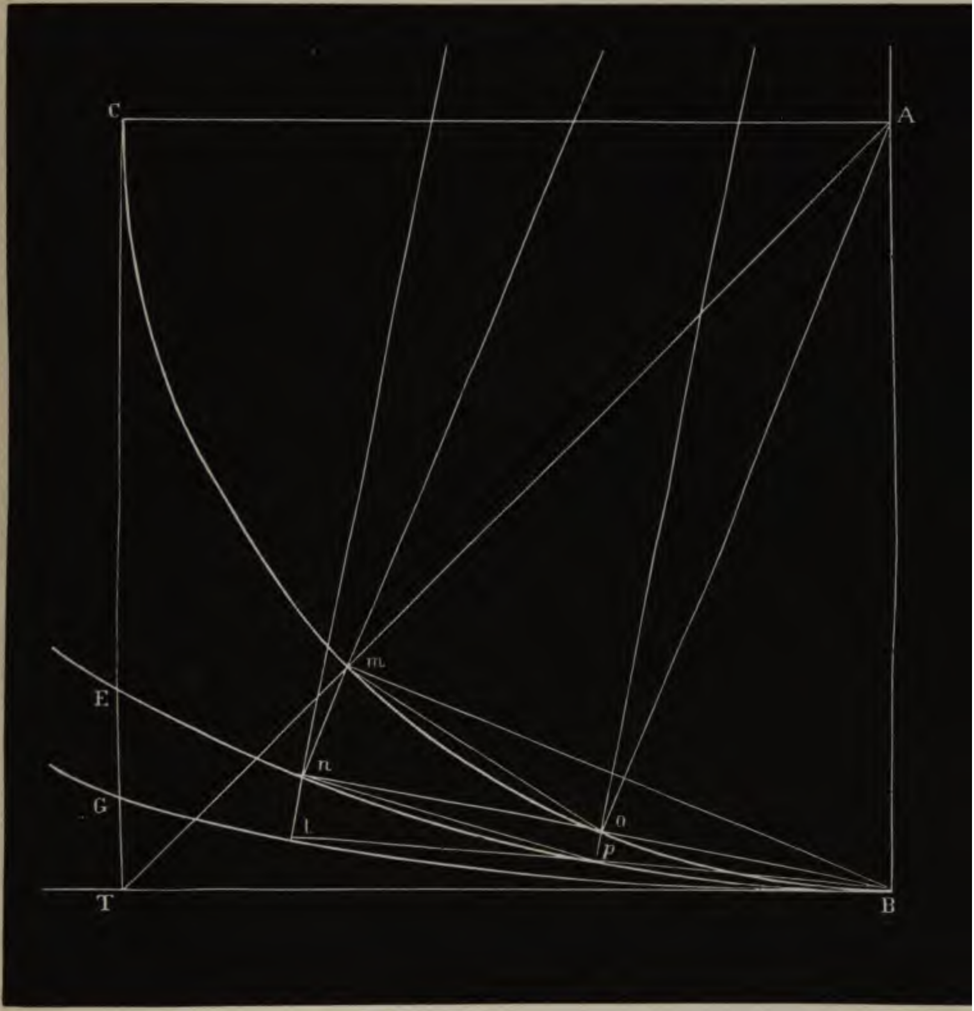
---

* Meaning the radius duplicated by doubling the distance of the centre with which the arc is described, from the point at the original extremity of the arc. (*i.e.* from the extremity of the radius joining the tangent.)

(3.) In like manner, a line is drawn through the point thus found at right angles to its chord, which line is a part of the secant belonging to the second half-arc (the half of the 1st duplicated half-arc) radially duplicated in magnitude. Now in order to draw the chord of the third duplicatod arc, without actually describing the second arc, the chord, or any part of the chord, of the second arc, may be taken as a radius and an arc described therewith intercepting the tangent of the primary arc; the arc so described being bisected, the point of bisection would be necessarily a point in the line drawn from the extremity of the perpendicular radius through the point of bisection of the first duplicated half-arc (if the arc were to be described and the line to be drawn.) And the line being drawn from the extremity of the perpendicular radius through the point thus found and produced until it intercepts the secant last drawn, the point of interception will be the terminal point of the second half-arc radially quadrupled in magnitude, and the line joining that point with the extremity of the radius is the chord belonging to the second half-arc quadrupled. In like manner the process may be carried on so long as any space remains between the tangent of the primary arc and the terminal extremity of the arc last drawn.

Fig. 18. This explanation may be made, perhaps, more readily appreciable by aid of the figure, which illustrates each step of the process just described; thus. .(1) The chord *B.m.* of the arc *B.m.* is drawn. Through the terminal point *m.* of the arc, a line of indefinite length, perpendicular to the chord *B.m.*, is drawn (of which line in the figure, *m.n.* is the extremity). (2) The arc *B.m.* is bisected in the point *o*: and through the point of bisection, from the extremity *B.*, of the perpendicular radius *A.B.*, a line is drawn which intercepts the line drawn through the point *m.* perpendicular to the chord *B.n.*, in the point *n.* The line terminating in

FIG. 18.



m.n. is a part of the radius belonging to the 1st duplicated half-arc, and the point *n.* shall be the terminal point of that arc: to wit, if the radius *A.B.* were to be doubled in length through the centre *A.*, and an octant *B.n.c.* were to be described with that duplicated radius, the point *n.* would be the point of bisection of that octant. (3) In like manner a line (of indefinite length) is drawn through the point *n.* perpendicular to the chord *B.o.n.* of the 1st duplicated half-arc (of which line, in the figure, *n.l.* is the extremity) is a part of the radius belonging to the 2nd duplicated half-arc.* Now, if we take *B.* as a centre, and the chord *B.o.n.*, or any part of *B.o.n.* as a radius and describe an arc terminated by interception of the tragential line *B.T.*, and bisect the arc so described. .that point of bisection will necessarily be a point in the chord *B.p.l.* of the 2nd duplicated half-arc, and if a line be drawn from *B.* through that point of bisection, that line will be the chord *B.p.l.* and will intercept the line drawn through the point *n.* perpendicular to the chord *B.o.n.*, in the point *l.* Also the line *B.p.l.* so drawn through the point of bisection of the (vertical) arc shall bisect the first duplicated half-arc in the point *p.*

In like manner the process may be carried on until the chord of the arc last drawn almost coincides with the tangential line *B.T.*, and the arc itself is (so to speak) projected upon that line, and which last arc represents a small fraction of the arc *B.M.* increased *radially* to the *magnitude* of the entire arc *B.M.*

The foregoing is a brief explanation of the general method which, however, admits of several variations by means of which the construction may be made to more completely exhibit and illustrate the application of the method to the purpose of measuring and investigating the lines pertaining to the successive arcs. Fig. 20, for example, in addition to the preceding explanation, which is illustrated therein, shows a convenient method of obtaining the terminal parts of the radii belonging to the successive arcs respectively.

---

* Evidently the 2nd duplicated half-arc *B.p.l.* is equivalent to the 2nd bisection of the original octant *B.m.*; *i.e.*, it is equivalent to the one-fourth part of the arc *B.m.*, radially quadrupled in magnitude.

In order to make the reliability and computative value of the method distinctly apparent, we will now, taking the radius equal to 10, and commencing with the half-quadrant, carry the computation through eight successive bisections, and radial duplications, and then compare the results with those obtained by Legendre from the process of inscribing and circumscribing polygons.

(*Note.* The square root of 'the square of the radius less the square of the sine' deducted from the radius, gives the versed sine.)

The square of the sine together with the square of the versed sine give the square of the chord.

The chord of (the half quadrant) $\frac{C}{8}$ equals the sine of $\frac{C}{16}$ duplicated; and so on.)

Radius = 10.

The half-quadrant R = 10	$\frac{C}{8}$	Sine	$7\cdot071068^2$	=	50·000000
		V.S.	$2\cdot92893^2$	=	8·578631
		Chord	$7\cdot653667^2$	=	58·578631
1st Bisectn. R = 20	$\frac{C}{16}$	Sine	$7\cdot653667^2$	=	58·578631
		V.S.	$1\cdot52241^2$	=	2·317732
		Chord	$7\cdot803612^2$	=	60·896363
2nd Bisectn R = 40	$\frac{C}{32}$	Sine	$7\cdot803612^2$	=	60·896363
		V.S.	$\cdot76859^2$	=	·590745
		Chord	$7\cdot84137^2$	=	61·487108
3rd Bisectn R = 80	$\frac{C}{64}$	Sine	$7\cdot84137^2$	=	61·487108
		V.S.	$\cdot38522^2$	=	.148394
		Chord	$7\cdot850827^2$	=	61·635502

4th Bisectn.	$\frac{C}{128}$	Sine	$7\cdot850827^2$	=	61·635502
R = 160		V.S.	$\cdot19271^2$	=	·037249
		Chord	$7\cdot85319^2$	=	61·672751
5th Bisectn.	$\frac{C}{256}$	Sine	$7\cdot85319^2$	=	61·672751
R = 320		V.S.	$.0964^2$	=	·009293
		Chord	$7\cdot85379^2$	=	61·682044
6th Bisectn.	$\frac{C}{512}$	Sine	$7\cdot85379^2$	=	61·682044
R = 640		V.S.	$\cdot0482^2$	=	·0023232
		Chord	$7\cdot85393^2$	=	61·6843672
7th Bisectn.	$\frac{C}{1024}$	Sine	$7\cdot85393^2$	=	61·6843672
R = 1280		V.S.	$\cdot0241^2$	=	·0005808
		Chord	$7\cdot85396^2$	=	61·6849480
8th Bisectn.	$\frac{C}{2048}$	Sine	$7\cdot85396^2$	=	61·6849480
R = 2560		V.S.			
		Chord			

To compare Legendre's computation * according to the method of inscribing and circumscribing polygons: since he takes the whole circle and the diameter as unity, his figures must be multiplied by 10 and divided by 4.

The inscribed polygon of Legendre's method; multiplying the figures by 10, and dividing by 4, to compare with the half quadrant and with Radius=10.

The bisected arc duplicated. The sine lengths belong to the successive arcs of bisection when increased in magnitude to equal the half quadrant.

The number of sides.		The lengths.	The lengths.	The fractional parts of the circle.
8	=	7·071068	7.071067	The 8th part.
16	=	7·653668	7·653667	16th "
32	=	7·803613	7·803612	32nd "
64	=	7·841371	7·841371	64th "
128	=	7·850828	7·850827	128th "
256	=	7·853193	7·85319	256th "
512	=	7·853785	7·85379	512th "
1024	=	7·853932	7·85393	1024th "
2048	=	7·853969	7·85396	2048th "

---

* The tabulated figures of Legendre's computation will be found at page 41.

It is sufficiently evident that the two methods of computation are precisely equivalent in their results. To correctly appreciate the true relationship and significance of that result it is necessary to particularly note that the radius of each successive arc is of increased magnitude; in such wise that the seventh arc, resulting from the process, is comparable with its radius which equals 1280; whereas the primary arc is comparable with its radius which equals 10. Therefore although the seventh arc is the same as the one hundred and twenty-eighth part of the primary arc radially increased 128 times, (and may be considered as equivalent to the primary arc from which the greater part of its curvature has been eliminated,) yet relatively to the radius and therefore relatively to the half-quadrant, the seventh arc is the 128th part of that primary arc divided off therefrom by repeated bisection, and, relatively to the tangential straight line, dissimilar from the primary arc in form. It now clearly appears, although the sum of the ultimate sines contained in the seventh arc, approximates to the length of that arc, that since 128 of thesefractional arcs must be combined in order to produce the half-quadrant, the difference between the sum of the ultimate sines contained in the seventh arc and the arc-length of the seventh arc, whatever that difference may be, is subject to multiplication by 128, in order to obtain the difference represented by the ratio of that ultimate sine-length when increased by 128 magnitudes, to the arc-length of the half-quadrant.

A practical advantage in this method over that of Legendre and others for the purpose of quantitive investigation is that the one continued computation determines the sine-lengths belonging to each of the successive arcs. The arcs are all equal in length each to each, and as the curvature is eliminated from the arc by the successive bisections and duplications of magnitude, the ratio of the sine length to the arc-length of the arc continually approaches equality, (*i.e.* until the ultimate limit at the vanishing point of the circle is reached.) Much facility in carrying on the computation arises from the fact, which an examination of the figure will make apparent, that the chord of the one arc is the sine of the arc next succeeding; for example, the chord of the primary arc of 45 degrees is the sine of the duplicated arc of 22½ degrees. The chord of the arc of 22½ degrees is the sine of the duplicated arc of 11¼ degrees, and so on. Hence it is to be also observed that the square of the chord length, by means of which the quantitive value of the versed sine to each successive arc is obtained, is continually furnished by the computation itself.

Figs. 22, 27, 28, 29, 30. (Part Third) may be referred to as illustrations of the process. Fig. 22 especially illustrates a modification of the process by which, without describing the primary octant or the successive arcs, the ultimate sine-length increased to the correspondent magnitude of the primary octant may be drawn and the arc-length of the octant may be cut off with a very close approximation to absolute accuracy, from a given straight line of indefinite length.* Figs. (1 and 2) 20 *R.* also illustrate this process.

---

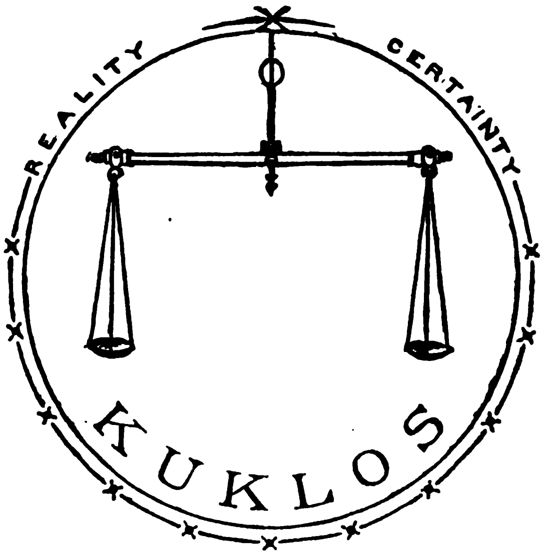
* Taking the radius of the primary octant as unity (*i.e.*, *A.B.*, the radius, = 1) the quantity of length so cut off will equal ·785397.... The quantity which has been erroneously supposed to represent the arc-length

ILLUSTRATION TO
Fig. 20 (R.)

THE CIRCLE

AND

STRAIGHT LINE.



PART THIRD

THE CIRCLE

AND

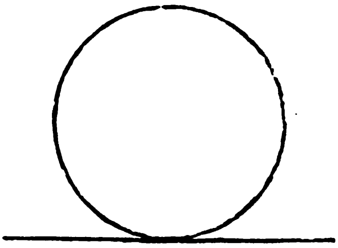
STRAIGHT LINE

PART THIRD.

ELEMENTS OF THE CIRCLE,

AND

THE THEORY OF CURVATURE.



BY

JOHN HARRIS.

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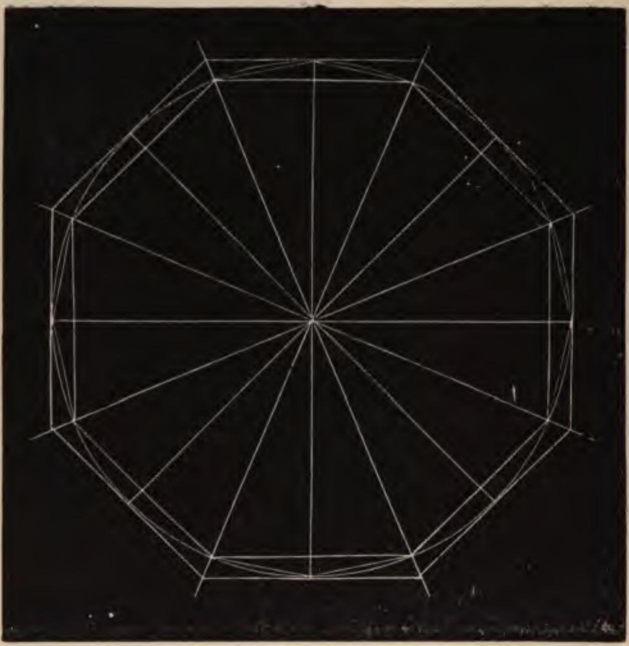

PREFACE TO PART THIRD.

In the Appendix to this part we have carried further and concluded the definition of the circle as a complete curvilinear boundary enclosing space. We have finally defined the boundary by exhibiting the relations of the three circles, *viz.*, the circle of the ultimate sine; the circle proper (*i.e.*, the true circumference of the circle); and the circle of the ultimate tangent. Thus showing not only that the perimeter of the circle as ordinarily considered is a breadth-containing boundary, but also the actual quantitive relation of that breadth to the length of the radius, whatever the magnitude of the circle may be.

The explanation may be summed up in this wise. If the circle proper be rolled upon the interior surface of the circle of the tangent, the circumference, *i.e.*, the outer surface of the former, will equal (coincide with) the interior surface of the latter. And the perimeter of the circle itself may be considered to have a breadth of which the measurement is 0·00017555 . . . taking the diameter as equalling 1. This breadth deducted from 0·5 gives the radius of the circle of the ultimate sine, which circle represents the inner surface of the perimeter of the true circle and measures 3·1415926 . . . whereas the circumference of the circle itself measures 3·1426964 . . and the outer surface of the circle of the ultimate tangent measures 3·1438002 . . .

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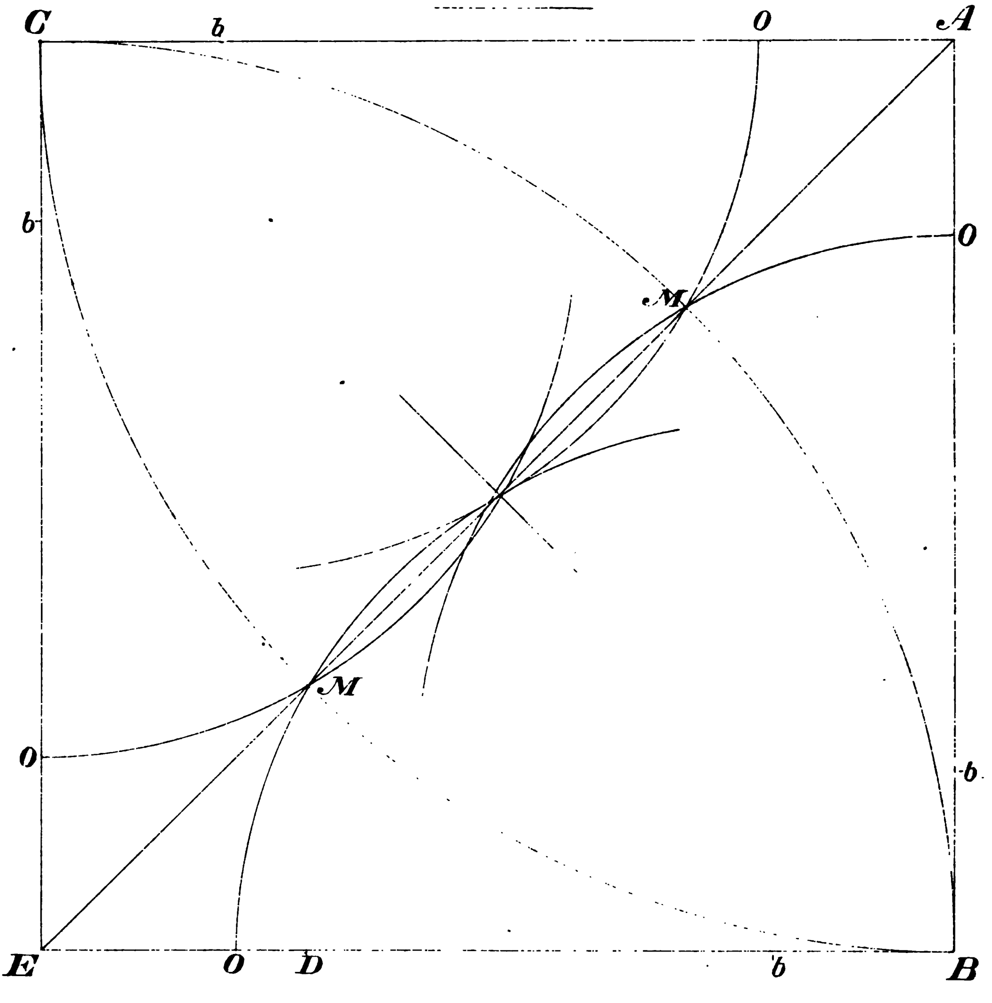
FIG. 37. R.



The Cyclal arc of decurvature

THE CIRCLE AND STRAIGHT LINE.

> 'Prove *all* things; hold fast *that* which is *good*.'
>
> *St. Paul.*

PART THIRD.

ELEMENTS OF THE CIRCLE.

The Ultimate-Sine, and the arc of increasing curvature.—If a number of intermediate arcs be described with a proportionally increased radius, between *M.* and *n.*, between *n.* and *o.*, between *o.* and *p.*, &c., &c., respectively, it is evident that a continuous curved line drawn through the terminal points of all those arcs, will form a compounded arc of a peculiar character and which from its forming a connection as it were between the arc-length of the half-quadrant and the ultimate sine—i.e. the sine of the ultimate fraction of the half-quadrant—possesses much interest. We will here briefly indicate one mode in which the change in longitudinal magnitude in consequence of the elimination of curvature, as the fractional arc is diminished by successive bisections, may be investigated.

Theory of Curvature.

Taking the radius equal to 10. The length of the half-quadrant equals the sine-length increased by addition of 'the sine-length divided by the units of radius less one.'

$$\frac{C}{8} = S + \frac{S}{R-1}$$

We thus obtain the magnitudinal length of the arc in units of radius. Now if this arc (half quadrant) be bisected and the longitudinal magnitude of the remaining arc be duplicated, the half-quadrant is not reproduced but an arc is obtained into which one-half the curvature of the half-quadrant does not enter and in which the other half of the curvature belonging to the half-quadrant is divided over twice the quantity of linear extension.* The second arc therefore, although it is a magnitudinal duplication of the one-half of the first arc, is a lesser longitudinal magnitude than the first arc, because the magnitudinal duplication which has restored the direct linear extension has not replaced the curvature.

The process may be thus conducted :—

$\frac{C.}{8}$ The arc-length equals the sine-length increased by addition of the sine divided by..the units in the radius less one.

$\frac{C.}{16}$ The arc-length equals the sine-length increased by addition of the sine divided by..the units in the (duplicated) radius less one, multiplied by two.

$\frac{C.}{32}$ The arc length equals the sine-length increased by addition of the sine divided by..the units in the (duplicated) radius less one, multiplied by two, multiplied by four.

$\frac{C.}{64}$ The arc-length equals the sine-length increased by addition of the sine divided by..the units in the (duplicated) radius less one, multiplied by two, multiplied by four, multiplied by four. And so on, proceeding in like manner.

The successive divisors will be therefore	1st	arc	$(R-1)$
	2	do	$(2 R-1) \times 2$
	3	do	$(2 R-1) \times 2 \times 4$
	4	do	$(2 R-1) \times 2 \times 4 \times 4$
	5	do	$(2 R-1) \times 2 \times 4 \times 4 \times 4$
	6	do	$(2 R-1) \times 2 \times 4 \times 4 \times 4 \times 4$

---

* If the half-quadrant, instead of bisected, were to be partially straightened into the same form as the half-arc, it would exceed in length the duplicated half-arc by the difference of the curvature. (i. e.) the difference between the curvature in the half quadrant and in the arc of 22½ degrees.)

Taking the successive sine-lengths from the table given at page 54, Part Second, we obtain :—

E	*A*		*B*		*C*	*D*
9	7·07107	+	·785674	=	7·856744	1 *R* — 1
38	7·65366	+	·201413	=	7·855080	4 *R* — 2
152	7·80361	+	·051339	=	7·854951	16 *R* — 8
608	7·84137	+	·012867	=	7·854597	64 *R* — 32
2432	7·85082	+	·0032281	=	7·854055	256 *R* — 128
9728	7·85319	+	·0008073	=	7·853997	1024 *R* — 512
38912	7·85379	+	·0002018	=	7·853991	4096 *R* — 2048
155648	7·85393	+	·0000504	=	7·853984	16384 *R* — 8192
622592	7·85396	+	·0000126	=	7·853978	65536 *R* —32768

In this table, the divisors in the first column *E*. result from the last column *D*. Under *A*. are the sine-lengths. Under *B*. are the quotients of the division of the sine-lengths by the numbers under *E*, which quotients added to the sine-lengths give the arc-lengths in column C.

It becomes evident that, by continuing the table, the quantities in column *B*., which represent the curvature, would become continually less, and the quantities under *A*. and *C*. would approximate more closely. In reference to column *B*. it may be observed that the third quotient is rather greater than the fourth part of the 2nd, that the 4th is a little more than the fourth part of the 3rd, the 5th a very little more than the fourth part of the 4th and that, of the remaining quotients, each is the fourth part of that preceding it, so far as the figures are carried ; a greater number of decimals would show each of the successive lesser quantities to slightly exceed the one-fourth of the quantity preceding it, but the approximation to the one-fourth becoming continually closer as the process is carried further. To appreciate the meaning of this it is necessary to remember that the chord of each arc, which chord equals the sine of the next succeeding arc, is the square root of the square of the versed sine added to the square of the sine; and that the versed sine belonging to each arc is always greater than the one-half of that belonging to the arc preceding it; but as the terminal radius approaches the perpendicular (i.e., as the curvature becomes more nearly eliminated), the versed sine of each arc approximates more and more closely to the one-half of that belonging to the arc preceding it.

The process thus suggested is put forward as worthy of attentive consideration, and for the purpose of illustrating the subject; but we wish it to be particularly observed that (for the present) it stands by itself, nothing else is based upon it; it is not put forward as a demonstration, nor can we (at present) vouch for the correctness of the process or for the accuracy of the figures. *See Appendix.*

THE ARC OF INCREASING CURVATURE—This arc is distinguished from the arc of the arc-length (that is from an arc described with the length of the arc as a radius), illustrated in Figs. 21 and 22. In Fig. 21, *𝒳.𝒳.* represents the arc described with the arc-length for a radius, *S. n. o. p.* the arc of increasing curvature. In Fig. 22, the relation of the same two arcs to the construction of the circle is illustrated on a larger scale.

(Note.—See Appendix, and with these Figures compare Figs. 20 *and* 25.)

THE TANGENTIAL LINE *B.D.*—

(Fig. 23 is a development of Fig. 12.)

Fig 23. Bisect the radius *A.B.* in *a.*, and with centre *a.*, and radius *a.B.* describe the quadrant *B.K.*; bisect the quadrant in *m.* and bisect the half-quadrant *B.m.* in *'n.*; through *'n.* draw *a.n.* intercepting the line *B.D.* at *n.* Bisect also *C.D.* in *b.*, and with radius *b.D.* describe the quadrant *D.K.;* bisect *D.K.* in *M.* and bisect the half quadrant *D.M.* in *'N.;* through *'N.* draw *b.N.* intercepting the line *B.D.* at *N.* And bisect the arc *B.S.* in *'p.*, and through *'p* draw *A.p.* With centre *C.* and radius *C.D.* describe *A.D.* bisected by the line *R.T.* in *Z.*, bisect *D.Z.* in *'P* and through *'P* draw *C.P.* intercepting *B.D.* at *P.*

We now have a division of the definite line *B.D.* into certain known quantitive magnitudes. Because *B.n.* is the tangent of the arc *B.'n.*, and *B.p.* is the tangent of the similar greater arc *B.'p.* which is twice the magnitude of *B.n.*, *B.p.* is twice the length of *B.n.*, and *n.p.* is equal to *B.n.* At the opposite extremity the similar points *D.N.* and *D.P.* are similarly related, and *N.P.* is equal to *D.N.*, which is equal to *B.n.*
Also because *B. q.* is (equivalent to) the sine of *B.m.*, *B.q.* is the half of *B.W.* which is (equivalent to) the sine of the similar greater arc *B.S.*
Taking the radius *A.B.* = 10, the numerical values will be:—

$B.K. = 5{\cdot}00000 = D.K.$
$B.n. = 2{\cdot}07107 = D.N. = n.p.$
$B.p. = 4{\cdot}14214 = D.P.$
$B.q. = 7{\cdot}07107 \div 2 = 3.535535 = B.Q.$
$P.K. = 5 - 4{\cdot}142136 = {\cdot}857864 = p.K.$
$B.N. = 10 - 2{\cdot}071068 = 7{\cdot}92893 = 3Bn + 2\ p.K.$

Having in mind the relative magnitudes of the primary lines determined by trigonometry, these values present themselves at once as evident; we may then obtain, from and by means of these, the quantitive values of other lines, and may also, thus quantitively, determine the geometrical (magnitudinal) relation of other lines; that is to say—the inductive reasoning, by means of which further knowledge of the relation of the parts of the circle to each other is to be obtained, may be based upon the facts belonging to the science of 'Number and Quantity' in place of those belonging to 'Form and Magnitude.'
For example:
Because $B.n. = 2{\cdot}07107$, $D.K = 5$, and $B.W. = 7{\cdot}07107$
Therefore $K.n. = W.D. = S.W. = S.T. = K.N. = 2{\cdot}92893$
And, $2{\cdot}92893 \times 7{\cdot}07107 = 20{\cdot}7107\ldots.$
Therefore, $S.W. \times B.W. = B.n. \times B.D$ That is—the versed sine × the sine of the half-quadrant = the tangent to the arc of 11¼°× the tangent to the half-quadrant.

Because $K.p. = B.K. - B.p. = (5 - 4{\cdot}142146) = {\cdot}857864$
therefore $K.p. \times 10 = 10\,K.p. = (K.n.)^2 = 2{\cdot}92893^2 = 8{\cdot}57864$
$= (S.T.)^2 = S.W.)^2 = (P.p. \times 5).$
And $P.p. = 2\,K.p. = 1{\cdot}71572$
But the square of $4{\cdot}142136 = 17{\cdot}1572$.
Therefore $P.p. \times 10 = (S.D.)^2 = (B.p.)^2 = (D.P.)^2$, &c., &c.
$B.P. = B.K. + K.P. = 5{\cdot}857864 = 2\,W.D. = 2K.n.$
Now the tangent to the half of the bisected half-quadrant duplicated, $= 8{\cdot}28428 = 2\,B.p. = 2\,D.P.$ And $(2B.p. - B.W.) = (K.n. - 2\,K.p.) = (8{\cdot}28428 - 7{\cdot}07107) = (2{\cdot}92893 - 1{\cdot}71572) = 1{\cdot}21321$, &c., &c.

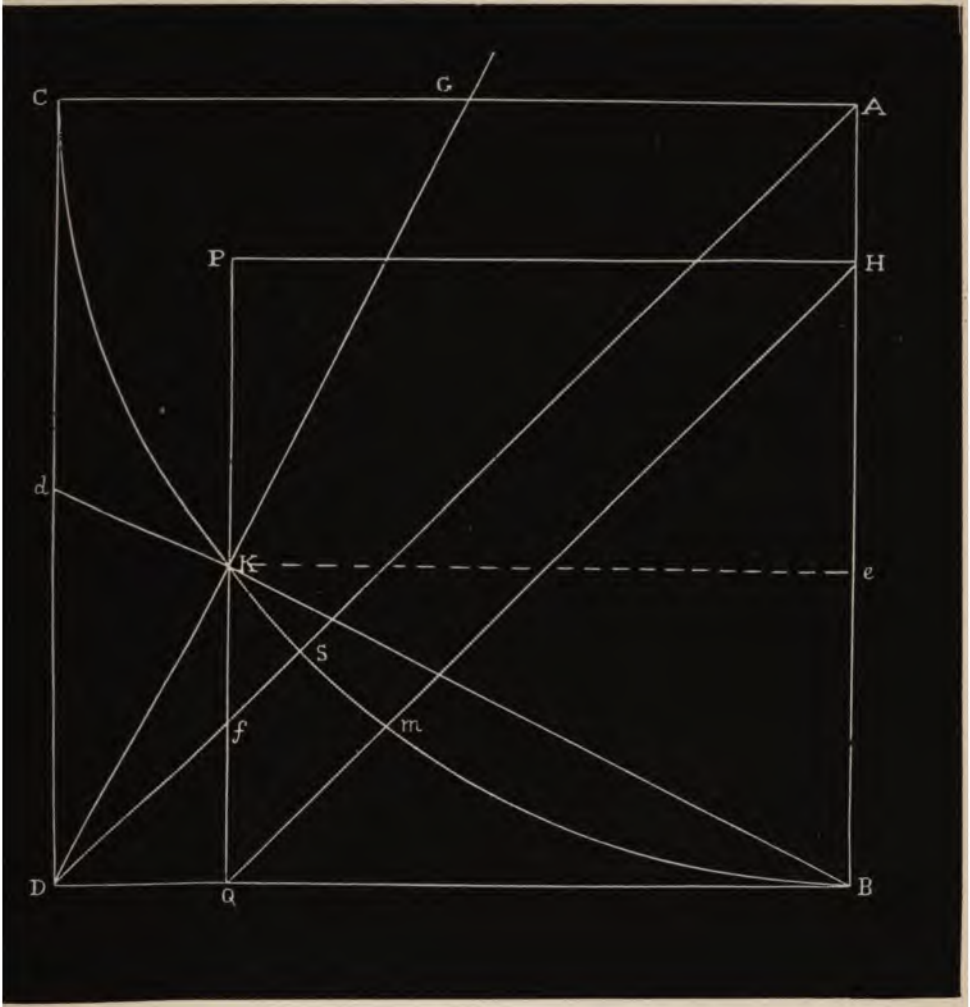
A fact of a primary character, which has not, we believe, been as yet applied in the art of computation, or otherwise utilized, is exhibited in the following :—

THEOREM.—If the quadrant of a circle be bisected and the secant to the half-quadrant be drawn through the point of bisection; and if the radius be produced through the centre of the circle and the produced radius be made twice the length of the original radius—that is, equal to the diameter of the circle—and a line be drawn joining the extremity of the produced radius and the extremity of the tangent; then shall the sine of the arc cut off from the quadrant by the line so drawn have to the tangent of the half-quadrant the ratio of four to five.

Fig. 24. With centre *A.* and radius *A.B.* describe the quadrant *B.C.*, and bisect *B.C.* in the point *S.* Draw the tangent *B.D.* and secant *A.D.* Join *D.C.* and *A.C.* Produce the radius *A.B.* through *A.* and make *E.B.* the production of *A.B.* double the length of *A.B.* Join *E.D.* intersecting the quadrant at *K.* and bisecting the line *A.C.* at *g.* Bisect the line *C.D.* in the point *d.* and join *B.d.* intersecting the quadrant at the same point *K.* From *K.* draw *K.Q.* perpendicular to *B.D.* and intercepting *B.D.* at *Q.* From *Q.* draw *Q.H.* parallel to *A.D.* and intercepting *A.B.* at the point *H.* From *H.* draw *H.P.* parallel to *A. C.* and equal in length to *B.Q.* Join *P.K.*

FIG. 34

FIG. 24.



Because *E.B.* is twice the length of *B.D.*, and *B.D.* twice the length of *d.D.* the triangle *B.d.D.* is similar to the triangle *E.D.B.* But the angles *B.D.d.* and *E.B.D.* are both of them right angles, therefore the line *B.d.* is at right angles to the line *E.D.* Now since *H.Q.* is parallel to *A.D.* and *H.P.* is parallel to *A.C.* the triangle *H.P.Q.*, is similar to the triangle *A.C.D.* And since the line *B.d.* bisects *C.D.*, the same line *B.d.* also bisects the line *P.Q.* in the point of intersection *K.* The line *B.K.* (part of *B.d.*) has been shown to be at right angles to the line *K.D.* (part of *E.D.*) therefore the triangle *K.D.Q.* is similar to the triangle *B.K.Q.*, and *B.K.Q.* is the half of *P.Q.* And *P.Q.* is (by the construction,) equal to *B.Q.* therefore *B.Q.* is equal to twice *K.Q.*, and *K.Q.* is equal to twice *D.Q.*; therefore the line *B.Q.* has the same ratio to the whole line *B.D.* as 4 : 5. Draw *K.e.* the sine of the arc *B.K.* *K.e* is manifestly equal to *B.Q.*

Corollary.—Hence if an octant be described with the radius *H.B.*, the tangent of that octant shall equal the sine of the arc *B.K.* (And radius *A.B*, is to radius *H.B.* : : 5 : 4.)

Prop. G.—Problem.—Upon a given straight line it is required to describe an arc containing 45 degrees, such that if the tangent of the arc be divided into five equal parts, four of those parts shall be together equal to the given straight line.

Fig. 24.—Let *B.Q.* be the given straight line, it is required to describe the arc upon *B.Q.* From the point *Q.*, perpendicular to *B.Q*, draw *Q.P.*, equal in length to *B.Q.*; and from *B.*, perpendicular to *B Q.*, draw *B.H.*, also equal to *B.Q.* Join *H.Q.* Bisect *P.Q.*, in the point *K.* and join *B.K.* Produce *B.Q.* through *Q*, and produce *B.H.* through *H.* (Through *K.*, at right angles to *B.K.* draw *K.D.* intercepting the production of *B.Q.*, at *D.* From *D.* parallel to *Q.H.** draw *D.A.*, intercepting the production of *B.H.* at *A*, With centre *A.*,

and radius *A.B.*, describe the quadrant *B.S.C.*, intersecting *D.A.* in the point *S.* *B.S.* shall be the required arc.

---

* Or, join *B.P.*, and from *D.* at right angles to *B.P.* draw *D.A.*, intercepting the production of *B.H.* at *A.*

The demonstration to this solution is substantially contained in the demonstration to the theorem.)

Bisect *K.Q.* and, through *f.* the point of bisection, draw *D.A.*, parallel to *Q.H.* intercepting the production of *B.Q.* at *D.*, and intercepting the production of *B.H.* at *A.* With centre *A*, and radius *A.B.* describe the quadrant *B.S.C.* bisected by the line *D.A.* in the point *S.* *B.S.* shall be the required arc.*

Because *D.Q.f.* is a right angle and *f.* bisects *K.Q.* *Q.D.* is equal to the one half of *K.Q.* But *K.* bisects *P.Q.*, and *P.Q.* equals *B.Q.*, therefore *Q.D.* is equal to the one fourth part of *B.Q.* Wherefore if *B.D.*, the tangent of the arc *B.S.*, be divided into five equal parts, four of those equal parts, shall together contain the given straight line. Q.E.D.

It will be convenient to give here the quantitive demonstration of the decimal relationship of the versed sine to the arc-length of the octant. The fact was prominently exhibited at the commencement of Part Second as a special theorem, which we now repeat:—

Theorem.—That if the radius of the half-quadrant be produced through the centre of the circle until it become equal to the versed sine (of the half-quadrant) multiplied by ten, and with this radius an arc be described terminated by a line drawn from the centre of the greater circle—*i.e.*, from the point at the (central) extremity of the radius—through the point at the extremity of the half-quadrant, the tangent to the arc last described (of greater magnitude), cut off by the line so drawn from the tangent of the half-quadrant, will be equal to the arc-length of the half-quadrant.

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*Note.—We have subsequently termed this arc for distinction, the octuple arc. Its elements and the importance of its relations to other factors of the circle will be particularly considered in Part Fourth, (which see).

Fig. 25.—Let *B.S.* be the half-quadrant of which *A.B.* is the radius; produce *A.B.* through *A.* and make *R.B.* equal to ten times *S.W.* From *R.* through *S.* draw *R.X.* intersecting *B.D.* in the point *X.* *B.X.* shall be equal to the arc *B.S.**

Demonstration. (Quantitive). The length of the sine *S.N.*, the arc-length of *B.S.*, the length of the radius *R.B.*, and of the versed sine *S.W.*, are known;
we have to obtain the length of *B.X.*

The radius, *A.B.* = 10.
" *v.* sine, *S.W.* = 2·92893.
" radius, *R.B.* = 10 *S.W* = 29·2893.
R.N. = *R.B.* – *S.W* = (29·2893-2·92893) = 26·3603.

Now the lesser triangle *S.W.X.* is similar to the greater triangle *R.N.S.*

Therefore *W.X.*: *N.S.* :: *S.W*: *R.N.*
W.X. : 7·07107 :: 2·92893 : 26·3603.
W.X. = ·78567 But *N.S.* + *W.X.* = *B.X.*
Therefore, 7·07107 + ·78567 = 7·85674.

And *B.S.* the arc-length also equals 7·85674.
Wherefore the tangent *B.X.*, of the arc of greater magnitude is equal to the arc length of the half-quadrant *B.S.* (Q.E.D.)

Corollary. Hence manifestly *W.X.* equals the difference of the sine and arc-length of the half-quadrant.

The Ultimate tangent.—We have objected to Legendre's circumscribed polygon--which eventually almost coincides with the sine—that it cannot be a simple continuous figure but must be fragmentary and compound; and we based this conclusion primarily and directly on the manifest necessity that a circumscribed polygon if continuous must be greater than the circle which it surrounds. We propose now to show that Legendre's circumscribed polygon is in fact compounded of the minute ultimate tangents separate from each other but arranged together in the form of a polygon. To show this we will first consider the question whether the tangent to an arc, if that arc be an indefinitely small fraction of the circle, is necessarily greater than the arc itself. It is established that the tangent is always greater than the arc by the evidence of trigonometry, thus:—Let *A.B.* be the arc, and *A.d.*

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* Or:—From *X.*, through *S*, draw *X.Y.R.*, intercepting the production of *B.A.*, at *R.* *R.B.* shall be ten times the length of *S.W.*

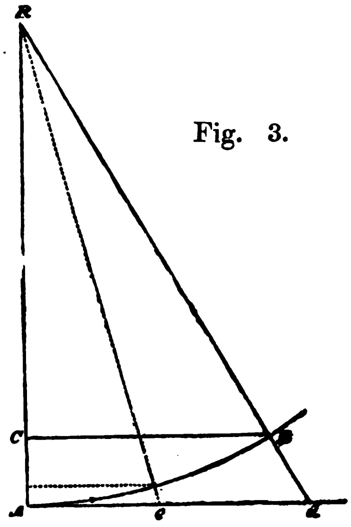

Fig. 3.

the tangent. The arc is bisected; and from the point *e*, *e.B.* is drawn perpendicular to *R.B.*, intercepting *R.B.* at *B.* Now the triangle *d.e.B.* is similar to the triangle *d.R.A.*, and the side *d.e.* is greater than the side *B.e.* of the lesser triangle, in the same ratio that the side *d.R.* is greater than the side *A.R.* of the greater triangle. Since evidently so long as it is possible to assign any quantity of magnitude to the remaining arc, it is possible to imagine the same process to be performed, it necessarily follows, that, however small a fraction of the circle an arc may be, so long as there be an arc, the same reasoning demonstrates the least tangent belonging thereto to be greater than the arc, because it is thereby shown that the outer tangent (as *A.d.*) is greater than the two interior tangents (as *A.e.* and *e.B.*) taken together; and it is quite manifest that the two interior tangents taken together are always greater than the arc.

Now Legendre does not deny, but indeed distinctly admits, that the tangent must be always a greater longitudinal magnitude than the fractional arc which it subtends, however far the reduction of the arc by bi-sectional division may be carried; but he argues that when the remaining section of the arc becomes extremely small the curvature is almost eliminated and the remainder of the tangent (or small side of the polygon) virtually (*i. e.* very nearly) coincides longitudinally with the remainder of the arc.

The fallacy in the application made by Legendre of the process adopted by him, may be said to consist in, or to arise from, the want of a complete apprehension as to the distinction between relative and absolute quantitive magnitude. All quantitve magnitude expressed as measurement is relative, and it is the expression of the ratio of one magnitude to another magnitude, which last is taken or given as the standard of comparison.

The object of Legendre's investigatiön is to ascertain the quantitive relation of the circumference of the circle to the diameter, expressed in units or standard increments of the diameter, or, which is equivalent, the relation of the octant to the radius expressed in units of radius. Legendre, increasing the number of sides of the circumscribed polygon by bi-sectional duplication, finally obtains a polygon of which one of the very small sides only very slightly exceeds in length the fractional arc which it subtends. He then measures the arc* and the tangent.. *i.e.,* he ascertains the comparative magnitudes of the small side and fractional arc, and the longitudinal value of the difference between them. How is that difference expressed? With what is the comparison made? Evidently with the diameter or radius of the circle circumscribed by the polygon; and is expressed in terms of that diameter or radius.

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* Strictly speaking he measures the sine and the tangent, and infers the comparative magnitude of the arc which is between them.

Legendre then, in order to obtain the quantitive relative magnitude of the complete polygon which circumscribes the circle, proceeds to multiply the quantitive contents of the one side by the number of the sides. Now the result of such proceeding is not what he supposes it to be, for the one side is connected, by magnitudinal relation, to the radius, in the units of which its quantitive value is expressed, and, consequently, by simply multiplying the side, he multiplies, together with it, the radius to which it is referred, and obtains, not a polygon of so many sides, but the one side of a polygon so many times greater..*i.e.*, of a polygon which would circumscribe a circle of so many times greater magnitude. In other words; the one side of the polygon is *radially increased*, but it remains the same small fraction of the circle which it was before; and so if the multiplication were to be repeated an indefinite number of times, the relationship would remain the same; if the small fractional arc subtended by the one side in the first instance was the one-thousandth part of the circumference, so would the fractional arc subtended by the enlarged side be the one-thousandth part of the circumference, but the circle in the last case might be a thousand or a million times the magnitude of the first circle.

Legendre admits that the small fraction of the circle ..say the one-thousandth part thereof..contains a difference between the two magnitudes representing respectively the fractional arc and the one side of the polygon. Is it not evident that a complete circle circumscribed by a complete polygon must contain one thousand times that amount of difference? Not, be it observed, one thousand times in reference to a radius one thousand times greater, because then the proportion would remain the same..the entire compound magnitude would be increased collectively, all the parts alike, and the quantitive ratio of each to the other would be unaltered, whereas what Legendre intends to do is to build up a complete polygon from the one side by the addition to that one side of all the other sides without increase or alteration of the radius. Let us see in what manner this requires to be done. We will suppose the complete circumference of the circle to be six times greater than the radius, and the one small equal side of the polygon to be slightly greater than the six thousandth part of the circumference..therefore very nearly equal to the one-thousandth part of the radius. And we will suppose the difference between the small side (tangent) and the fractional arc which it subtends, to equal the six-millionth part of the small side (tangent). By multiplying the six-millionth by the six thousand we obtain the ratio of the difference contained by the complete circle between the circumference and the circumscribed polygon on the scale of the one-thousandth of the radius. That difference would be therefore equal to the one-thousandth part of the small side (tangent).* The scale has then to be increased to the magnitude of the original circle and its radius, which will be done by multiplying the quantitive values of the small circle and its circumscribed polygon, each by six thousand. The result will be the quantitive values of the (original) circle and of its circumscribed polygon..the difference between which will be equal to the one-thousandth of that polygon

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* The small fractional arc is supposed to be divided into six thousand *similar* arcs. Now, to do this the original radius must be divided into six thousand parts..then with one *of these divisional parts as a radius* the complete circle is described, and the small polygon is circumscribed about it by dividing the small side (tangent) into six thousand equal parts, and taking one of the divisional parts for each side of the polygon. It is evident that the former ratio of the difference..to wit, the six millionth part of the small side (tangent) relatively to the original radius..must be, relatively to the reduced radius, increased proportionally to the reduction of the radius: must be increased therefore six thousand times.

(which polygon will contain six thousand sides). But by Legendre's mode of computing, the result would be a difference of only the six-millionth part of the same polygon.

Let *D.E.A.F.B.* (Fig. *a*.) be four equal divisional parts of the half-quadrant *D.B.* Fig. *b*. represents one of those equal divisional parts on a larger scale, the straight line being equal to *A.e.* Now if we apply four straight lines each equal to *A.e.* as tangents to the divisions of the half-quadrant, as shown in Fig. *a*. it is evident that they cannot meet each other at the extremities which are not in contact with the circle. If they had sufficient length to meet each other, angles would be formed at the points *E.* and *F.*, but, the length being insufficient, spaces are left at those points. Supposing the four lines to be again divided by bisection, and the eight half-lines to be arranged as tangents to the circle, there will be four spaces instead of two where the extremities of the lines will not quite meet each other, but they will be much more nearly in contact than before. If the lines be in like manner repeatedly divided, there will be eventually obtained a fragmentary polygon, formed by the ultimate tangents which will then be very nearly in contact. Let us now suppose that the lines were at first a very little longer than *A.e.* so that the ultimate divisional parts have sufficient length to become actually in contact; the extremities of each with the extremities of those next thereto. What are the conditions of that contact? If we accepted the dogma of Euclid, we should have to consider the contact as complete and the polygon as a simple figure, and to accept the sum of the ultimate tangent-lengths as the length of the circumscribed polygon, but we reject the dogma and insist on a real circle containing area and, consequently, containing breadth.

We have, accordingly, to enquire how the lines are divided. Is the section oblique or perpendicular to the

THE TANGENT.

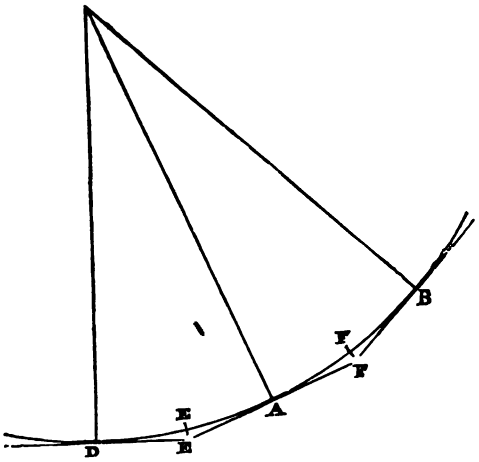


Fig. a.

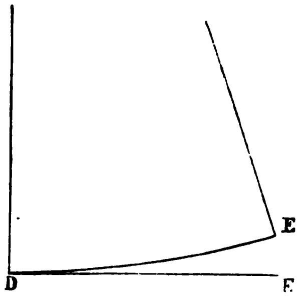


Fig. b.

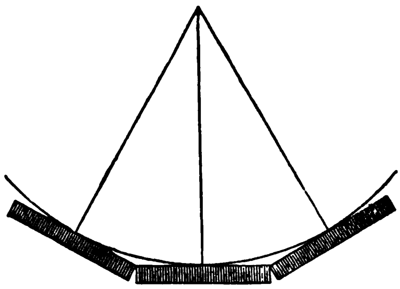


Fig. c.

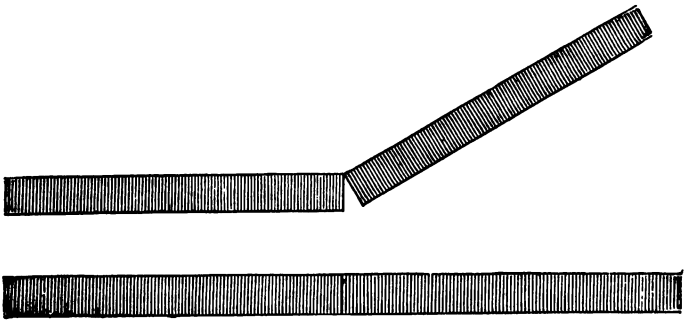


Fig. d.

length of the line? Now this is readily answered—because the divisional lines, derived from continued bisection, are divisions of a straight line evenly divided at right angles to its length. Therefore, by placing the divisional lines of Fig. *a.* in the positions, relative to each other, which they would occupy as tangents to the circle, and considering the rectangular extremities of the one with reference to the rectangular extremities of the next, the actual conditions of the contact between them will become manifest. Figs. *c.* and *d.*

The fragmentary tangent-lines compounding the circumscribed polygon formed by piecing together the sections of a straight line divided at right angles to its length, are, therefore, in contact with respect to the inner surface only; but if a straight line be compounded of the fragmentary lines, the contact will be complete —that is, perfect with respect to both surfaces—and the original straight line will be restored.

This case may be taken by itself. The fact of the ultimate arc very nearly coincident in length with the ultimate sine and tangent, and of the half-quadrant very appreciably less than the circumscribed polygon and very appreciably greater than the sum of all its ultimate sines, presents itself for acceptance. That it is indisputable and must be accepted seems to us evident. But what then: if there be some persons to whom this fact appears irreconcilable with the facts (as now taught) of trigonometry?

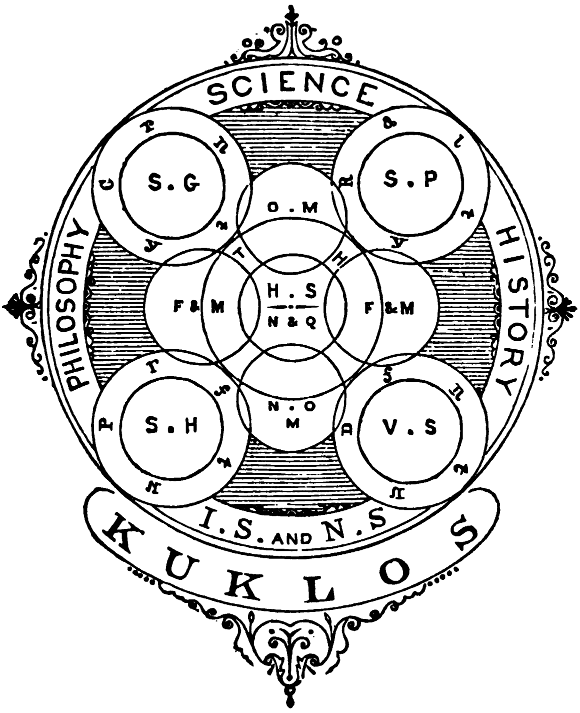
In concluding this treatise it may be not out of place to consider a little the meaning of the familiar statement (phrase) that one fact cannot contradict another. Probably no one at the present time with any pretension to scientific knowledge or education will directly deny or express any doubt as to the truth and certainty of such statement in its simple form; but the question is as to absolute certainty in the strictest and most comprehensive sense; and in the writings of men of scientific reputation, at the present time, opinions are expressed, theories propounded, and statements made, which, if referred inferentially to their basis, make apparent that, in the minds of the writers, no clear and reasonable conviction has been arrived at, as scientific certainty, that one fact may not or cannot contradict another; i.e., be otherwise than in harmony with other facts. We call attention here to this circumstance because we believe that, in many cases, if the writer distinctly understood the nature of the conclusions involved or of the reasonable inferences belonging as corollaries to those theoretical opinions and statements, they would be reconsidered and, being found to be unreasonable, would not be put forth. This circumstance, which must needs be regarded as of the gravest importance, may, in part at least, be a consequence of the increasing neglect of mathematical reasoning—by which we mean, strictly lawful and (therefore) correct reasoning—as a necessary element of educational training. A consequence of the want of such training is a non-recognition by the mind of the necessity of a primary basis, either securely established or undoubtingly believed in, upon which the theory or statement must be actually or presumably based. Practically the effect displays itself in the opinion now more or less openly expressed by scientific teachers that Theology, is not an absolutely necessary and fundamental part of Science but that, on the contrary, Science and Theology, if not actually antagonistic, should be kept quite distinct and separate.

This separation of Science from Theology appears to be concurred in, and even supported, by very many of those who teach that the fundamental basis of Science must be found in Theology. They, nevertheless, accept the title of theologians, as distinguished from, and as in some degree antagonistic to that of, men of Science. Moreover the dislike of the man of Science to Theology, is perhaps, more than reciprocated by the theologian whose belief in the primary truth of Theology is oftentimes a belief fearful and apprehensive lest the progress of Science should discover and make known some adverse truth irreconcilable with the truth of Theology.

That this view of the case is adopted by all or by nearly all scientific men or theologians we are not supposing. The exceptions on both sides are probably in the aggregate very many, but the question is, taken all together, what is the conclusion or state of feeling now entertained and held by the educated public generally; and in what direction are the doctrines now taught guiding those who are to succeed us?

Assuming the existing state of things to be substantially as we have suggested, the apprehensive belief of the theologian which fears the facts and discoveries of science is not a belief in a scientific sense; such a fear evidently includes a doubt as to whether one fact may not contradict and be irreconcilable with another or with some other facts. Nor can it help matters to say that the truth of theology is different from the truth of science, because, if it be meant thereby that the truth of the one can be otherwise than in perfect harmony with the truth of the other, the entertaining or admitting in any degree such supposition is the making over Theology to Metaphysics which is expressly the antagonist of sound Theology as being the essential basis of all sound Science. How stands the case on the other side? If the question was put: what is the especial principle contended for and upheld by modern science? the answer most generally concurred in would be—'intellectual freedom,' the right of each one to exercise his intellect in his own way, without let or hindrance. What does that mean? It means *intellectual lawlessness;* intellectual anarchy; the right of each one in matters which concern others as well as himself in the most important and momentous sense, to do as seemeth good in his own eyes; the demand of men, who recognise the imperious necessity of laws and submission to those laws in matters of comparatively little importance, to be free from the restraint of all law in matters of the greatest importance. It means the claim of some men to be allowed to impede sound education, to pollute things sacred, to contaminate all true knowledge, to teach the most pernicious doctrines, and to deceive and confound those who look to them for instruction.

Against this claim of so-called intellectual freedom, whether it be hesitatingly expressed, merely as a dislike to mixing up theology with science, or boldly and defiantly asserted as a determination to be intellectually unrestrained by any laws or rules, we distinctly protest and earnestly caution those whom we may in any degree influence. It may be said—let public discussion and progressive education put right what is wrong, we cannot expect to have science quite perfect at any time; that which is true is stronger than that which is false—increased knowledge and discussion may be relied upon to eliminate error. No: such reasoning is not itself based on fact; and such a course is not to be relied on to eliminate error; on the contrary if allowed to proceed, its effect would be to vitiate education and to deteriorate knowledge more and more, until, at length, the education would become an altogether evil education, and the knowledge an essentially unsound and destructive knowledge. Truth is stronger, much stronger, than falsehood, but a mixture of that which is true and that which is false is not truth. Neither is it fact that error can be eliminated and got rid of by discussion in which sound and unsound knowledge are mingled together, in which the reasoning is not strictly lawful and not properly based on that which is true and certain. Such discussion is the weapon with which Untruth, wearing the mask of Science, can best succeed in destroying that which is true, and confounding those who love the truth. A human science which does not distinctly recognize the primary truths of theology as its ultimate basis, is not based on reality; it has not and cannot have any actual and secure foundation. If the Science of England is not so based, no matter what seeming progress may for a time be made, whenever the trial comes it will be as the house built on the shifting sand; and, if not destroyed by sudden catastrophe, will eventually become a ruin, together with the civilization which rests upon it.



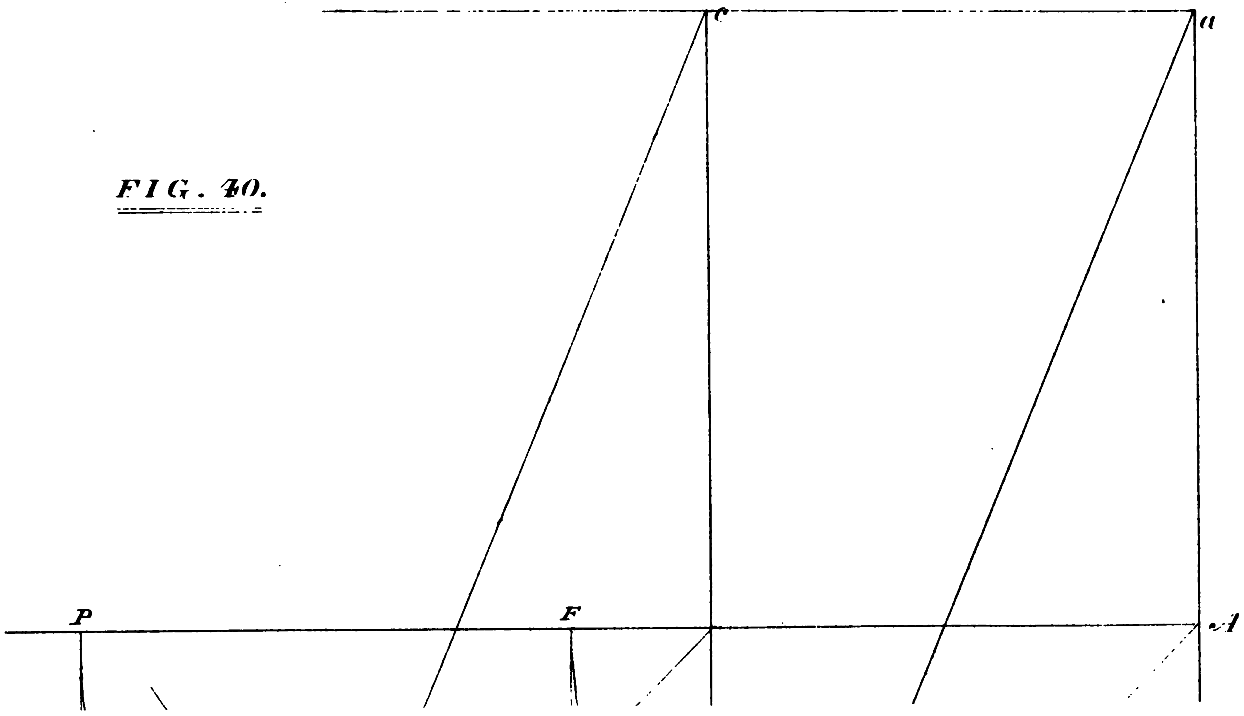
APPENDIX.

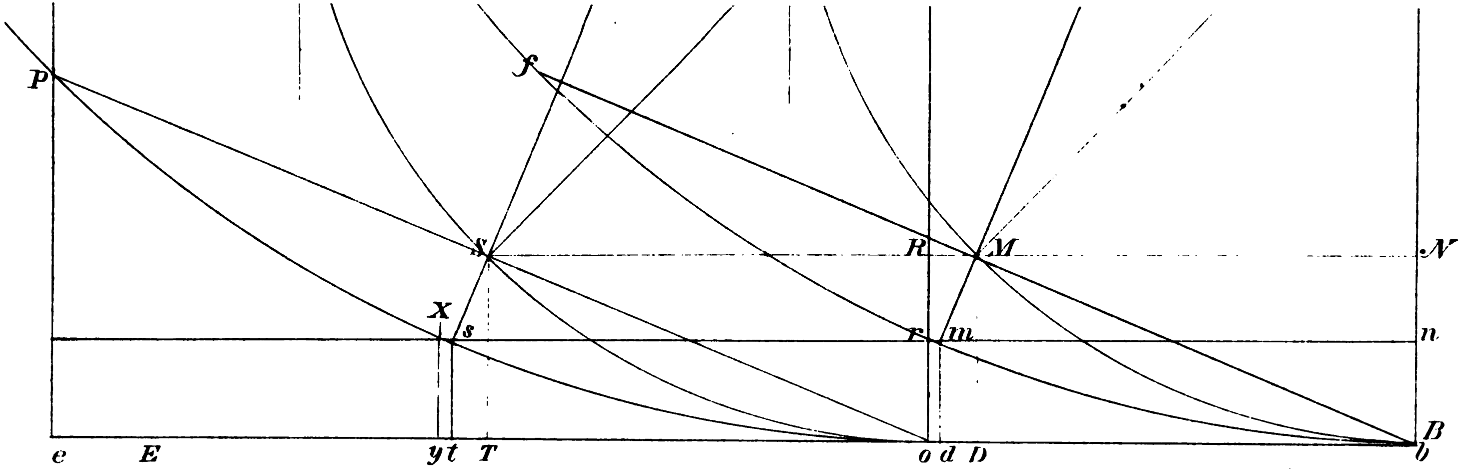
TO PART THIRD.

Demonstration that the Theory of curvature, as set forth at the commencement of this Part, is sound and represents the actual relation of the ultimate sine to the arc of the circle..may be exhibited by means of a modification of the fundamental construction. In fig. 40, the arc of 22½ degrees (*i. e.* $\frac{C}{16}$ or half octant) takes the place of the octant *B.M.*, belonging to figs. 1 and 2.—A little consideration will show that the same reasoning which was applied in Part First, to the preliminary examination and demonstration of the fundamental construction, will likewise apply to this case of the half-octant. Because the point *m.* at the terminal extremity of the arc (half-octant) *b.m.*, in its first position, becomes, when the arc is rolled on the tangential line *b.e.* in contact at the point *o.*, that point of contact must necessarily fall vertically beneath the point *r.* at the commencement (the original extremity) of the sine of the second half-octant *o.s.* And since, if the rolling be continued, the point *f.* at the terminal extremity of the second half-octant *m.f.* necessarily arrives at *s.* therefore the difference *r.m.* between the arc-length *m.s.* and the sine *r.s.* is demonstrated to be an aliquot part of the arc-length, and, consequently, also of the sine. It may be then shown, in like manner to the previous case of the octant (Part First), that the number of divisional parts, each equal to the diff. must be 39 in the arc-length and 38 in the sine.

It will be seen to follow that this aliquot relation of the diff. to the arc-length is a property which must be-

FIG. 40.





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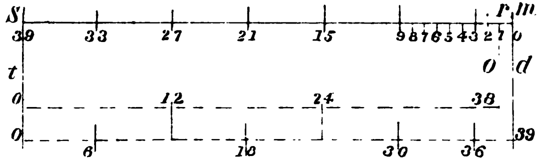

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long, not peculiarly to the octant and half-octant only, but (1) to every fractional arc resulting from the continued bisection, or even division of the octant, and (2) to every divisional part of the octant, whether even or uneven. Therefore, we may be said to arrive herein at the ultimate analysis of the relationship of the sine to the arc-length of the octant..for it becomes evident that in every arc resulting from the removal of a part of the octant, the diff. of the arc-length and sine in that diminished arc, compared with the diff. in the octant, is, in the first place, *proportionally* less by the increase in the sine-length (*i.e.*, the increase in the sine is a direct and equal diminution of the diff.) and, secondly that the remaining part of the diff. must be proportional to the (diminished) versed sine belonging to the reduced arc, the magnitude of which is evidently related to the specific curvature contained in the arc.

NOTE.—It would appear that the definite structural relationship of the sine to the arc of the circle, thus shown, must be a part of or closely related to the basis of the ideal science *Harmony*; and since the subjects of those divisions known as *Optics* and *Acoustics* belong in part to the science of Harmony, it is at least probable that the knowledge and appreciation of this relationship will lead to fuller apprehension of the laws belonging to those divisions.

Figs. 27, 28, 29, 30, are additional illustrations of the method of eliminating the curvature by continued radial duplication of the bisected arc.

It may be observed that in some of the figures the same letter of reference, *X.*, indicates the arc-length and the ultimate sine-length. On the scale of these figures the two points fall so closely together as to appear almost identical, the difference being only about three parts in eight thousand, and an attempt to show them separately in the same figure would confuse the lines and letters. In Fig. 39, however, *B. O.* indicates the arc-length, and *B.P.* the ultimate sine-length.

We have brought under consideration four arcs, namely:

(1) The arc of the arc-length *O.O.* (*𝔛.𝔛.* in Fig. 21.)

(2) The cyclal-arc of decurvature (or, of increasing curvature) *M.O.* (*S.𝔛.*)

(3) The arc of the ultimate sine-length *P.P.* (*𝔛.𝔛.*)

(4) The cyclal-arc of the increasing sine *M.P.* (*S.𝔛.*)

The first and third, indicated in the Figures by the same letter *𝔛.𝔛.*, because too close to each other to be clearly distinguishable, are both of them arcs of circles described from the same centre *B.* The radius of the one (1) equalling 7·85674..&c.; and the radius of the other (3) equalling 7·85398..&c.

The second and fourth arcs we wish now again to compare and contrast, in order to distinguish between them. In order to do this, let us first state the following lemma:

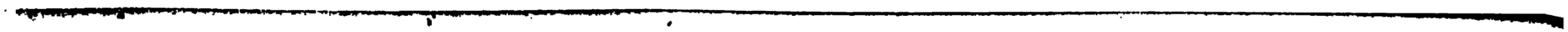
Lemma.—If a triangle be isosceles or equilateral, and if the poiut at the vertex of the triangle being taken as a centre, and one of the sides being taken as a radius, a circle be described therewith, then shall the points at the extremities of both sides be in the circumference of the circle, and the base of the triangle shall be the chord of the arc contained between the points at the extremities of the two sides.

The Cyclal Arc of decurvature, Fig. 37. If, therefore, the line *M.O.* (equivalent to *S.𝔛.* the extremity of the line *R.Y.𝔛.* Figs. 21 and 25), which is the chord of the arc of decurvature *M.O.*, be bisected, and from the point of bisection *d*, at right angles to the chord, a line (*d.b.*) be drawn intercepting the primary tangential line *B.D.*, the point of interception will be *b.* at a distance from *B.* = 2·000452...And the point *b.* shall be the centre, and the line *b.d.* the radius, of the arc of decurvature *M.O.*

The arc *O.M.* is a very little greater than 30 degrees.

The tangent *O.𝔛.* to the arc = 3·3831..

The sine *D.M.*, is also the versed sine of the primary



The Cyclal arc of decurvature

octant *B.M.* (to which the arc *O.M.* is cyclal), and equals 2·92893..

The Cyclal arc of the Increasing Sine.—Fig. 38.

By the same lemma, if a line be drawn from the extremity of the octant to the extremity *P.* of the ultimate sine-length on the line *B.D.*, then, if this line which is the chord of the cyclal arc of the ultimate sine be bisected, and from the point *c.* of bisection, perpendicular to the chord, a line *c.a.* be drawn intercepting the line *B.D.*, the point of interception *a.* shall be the centre and the line *c a.* the radius of the cyclal arc of the ultimate sine *P.M.* The arc *P.M.* is a little less than 30 degrees. The tangent *P.X.* to the arc = 3·37317... The sine *D.M.* is also the versed sine of the primary octant *B.M.* (to which the ultimate sine belongs and to which the arc *P.M.* of the ultimate sine is cyclal) and equals 2·92893...

Herein we observe a very interesting definite relationship established between the arcs *O.M.* and *P.M.* respectively, and the fractional arcs of the primary circle to which the octant *B.M.* belongs[*]; namely, the ratio of the circles to which they respectively belong is :—*O.M.* : *B.M.* : : radius *M.b.* : radius *A.B.* And, *P.M.* : *B.M.* : : radius *M.a.* : radius *A.B.*

Note.—Referring to Fig. 21 and 25, the following will be the elements of the computation :—Taking *X.* in the line *B.D.* to indicate the extremity of the arc-length (*i.e.*, *B.X.* = 7·85674), then, bisecting the chord *S.X.* and drawing the line *d.b.* intercepting *B.D.* at *b.*, we obtain *X.d.* as the base of a triangle similar to the triangle *S.X.W.*; therefore as *X.W.* : *X.S.* : : *X.d.* : *X.b.*

---

[*] That is ... Between the cyclal arcs and the primary arc to which they are cyclal. *Note.*—In Fig. 25 the point *M.* at the extremity of the octant is indicated by the letter *S.*, and the points *O.* and *P.* which fall very close to each other are both indicated by the letter *x.*

Now since *S.W.* = 2·92893.. *S.𝔛.* = 3·03247..
And *𝔛.d.* = *S.𝔛.* ÷ 2 = 1·516235, therefore
·785674 : 3·03247 :: 1·516235 : 5·85223

which last number is the length of the radius *b.𝔛.*

7·85674 − 5·85222 = 2·00452 = *B.b.*
But 7·07107 − 2·00452 = 5·06655
And $\sqrt{5{\cdot}06655^2+2{\cdot}92893^2}=5{\cdot}85223$

that is, the square of 'the sine of the half-quadrant less *B.b.*', added to the square of the versed sine *S.W.* gives the square of the radius *b.𝔛.*

Now taking *𝔛.*, in the line *B.D.*, to indicate the extremity of the ultimate sine-length radially increased, (*i.e.*, *B.𝔛.* = 7·85397...). The chord *𝔛.S.* is bisected in the point *c.* The base *𝔛.W.*, of the lesser triangle *S.𝔛.W.* now equals 7·8291...

And, since *S.W.* = 2·92893... *S.𝔛.* = 3·03176... *𝔛.c.* = *S.𝔛.* ÷ 2 = 1·51588... Therefore... 7·8291. : 3·03176 :: 1·51588 :: 5·87013... which last quantity is the length of the radius *a.𝔛.* And 7·85397 − 5·87012 = 1·98384 = *B.a.*

The following table exhibits a comparison of the sine-lengths throughout eight successive bisections of the octant.

A. are the sine-lengths of the successive arcs—obtained by the radial duplication of the successive fractions of the repeatedly bisected arc—by the terminal extremities of which 'the cyclal arc of the increasing sine' is formed.*

B. are the sine-lengths of the equivalent successive arcs (corresponding to those of column *A.*)—resulting from the continual 'unbending' or 'straightening' † the primary arc—by the terminal extremities of which 'the cyclal arc of decurvature' is formed.

---

* These sine-lengths agree with those of the table at page 55 of Appendix, Part Second. Instead, however, of bisection and radial duplication, the repeated diminution of the arc by successive removals of *any* fraction and the *proportional radial* increase of the remaining arcs may be substituted.

† The expression..decurving..might be preferable to that of..unbending..an arc, and would be, perhaps, more generally acceptable.

	A.	*B.*
$\frac{C}{8}$	7·071068	7·071068
$\frac{C}{16}$	7·653668	7·65517
$\frac{C}{32}$	7·803613	7 80606
$\frac{C}{64}$	7.841371	7·84405
$\frac{C}{128}$	7·850828	7.85356
$\frac{C}{256}$	7·853193	7.85539
$\frac{C}{512}$	7·853785	7·85655
$\frac{C}{1024}$	7.853937	7·85670
$\frac{C}{2048}$	7·853969	7.85673
Finally	7·85398 ...&c.	7.85674...&c.

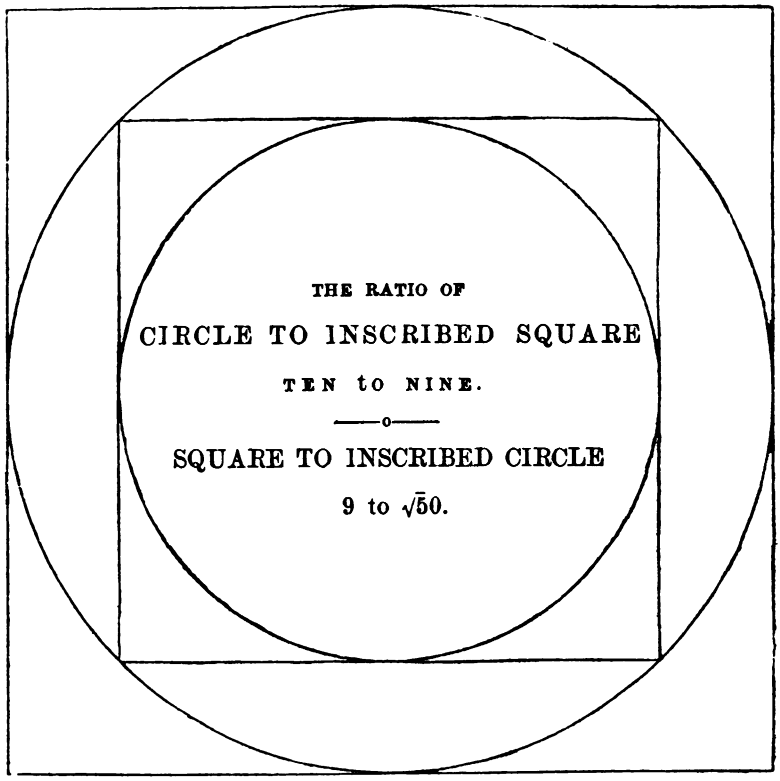
Referring to the explanation given in the supplement of Part Second of the method by which the ratio of the circumscribed polygon to the complete circle may be computed from the fractional arc and the fragment of the polygon..It is evident, since the lesser polygon is greater than the complete circle (described with the six-thousandth divisional part of the greater radius) about which it is circumscribed, that a circle described from the same centre as the lesser circle, equal in longitudinal magnitude to the polygon, must be described with a radius longer than the radius of the lesser circle in the same proportion as the polygon is greater than the lesser circle. Then if we increase the radius of the lesser circle six thousand times, and increase the radius of the circle equal to the lesser polygon, six-thousand times, we shall then obtain the radius of the larger circle and the radius of a circle which will equal the polygon circumscribed about the larger circle (*i.e.* will equal Legendre's circumscribed polygon). The circle thus obtained we will call for distinction, The circle of the ultimate tangent. We have now, therefore, three circles, to wit.... (1) The circle of the ultimate sine, equivalent to the inscribed polygon of Legendre. (2) The (true) circle itself. And (3) The circle of the ultimate tangent, equivalent to the circumscribed polygon of Legendre.

The explanation of the case may be now thus stated: The perimeter of the circle is a breadth-containing boundary of which the breadth measures 0.00017555..taking the diameter = 1. This breadth deducted from 0.5 gives the radius of the circle of the ultimate sine, which circle measuring 3.14159..represents the inner surface of the (breadth-containing) perimeter of the true circle, and of the (true) circle the circumference or outer surface measures 3.142696..and the breadth of the perimeter equals 0.00017555..Or the surface of the circle of the ultimate tangent may, for the sake of illustration, be considered as the outer surface of the perimeter, and the surface of the circle of the ultimate sine as the inner surface of the perimeter (the true circle occupying the mean distance between the inner and outer surface), in which case..the breadth of the perimeter will equal 0·0003511...to the radius as a unit..and the outer surface of the perimeter will measure 3·1438002...

Accordingly, the diameter of the three circles, respectively are: — (1) 0.9996489... (2) 1.0000... and 3) 1.00035122..

Note.—It is to be observed that the arcs which we have termed respectively .. The arc of the arc-length: The arc of decurvature; The arc of the ultimate sine-length; and the arc of the increasing sine; are all of them cyclal arcs—*i.e.*, cyclal to the octant of the primary—having the respective centres of their circles situated in the tangential line of the octant. The three circles of the perimeter are, of course, all of them primary, and have the one centre in common.

Having thus obtained the ratio of the intervening circle to the circumscribed and incribed polygon respectively, when the number of the sides of each of the two polygons is indefinitely great... let us now define the relation of the circle to the circumscribed and to the inscribed square, as demonstrated in the foregoing both by magnitudinal and quantitive geometry:—



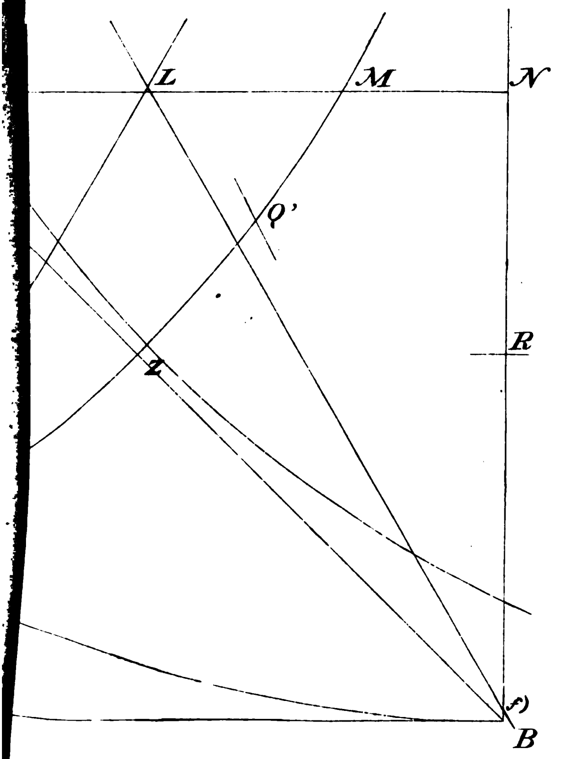
Therefore The Ratio of { *Circumscribed Circle to Inscribed Circle is* 10 : $\sqrt{50}$.
Circumscribed Square to Inscribed Square is 10 : $\sqrt{50}$.

THE CIRCLE

AND

STRAIGHT LINE.

PART FOURTH.

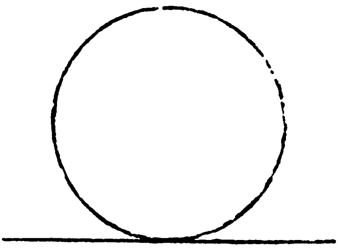


angles, which are parts thereof,
quilateral, &c., &c.

quilateral.

THE CIRCLE
AND
STRAIGHT LINE

PART FOURTH.
ANALYSIS OF THE CIRCLE.
AND
DECIMAL CYCLOMETRY.



BY
JOHN HARRIS.

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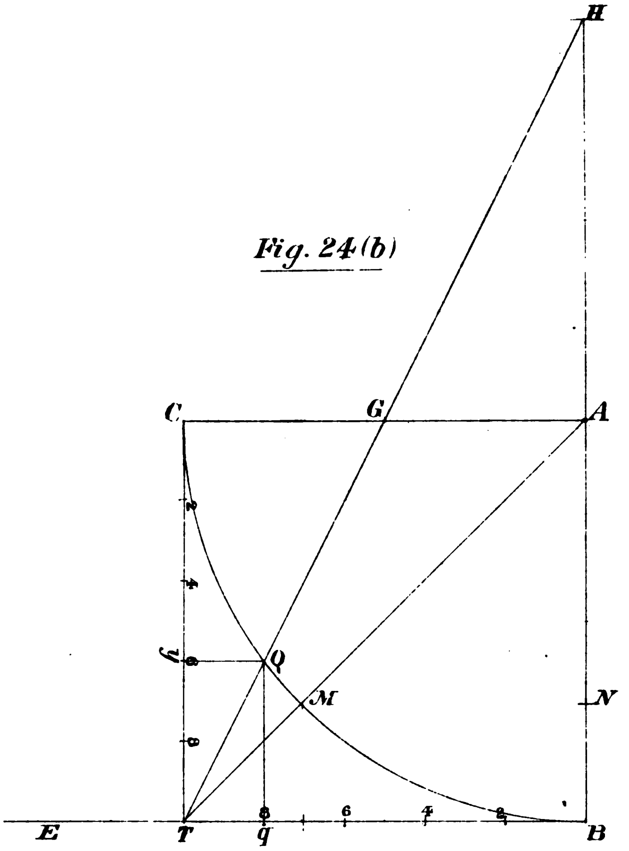
Fig. 24(b)



The Octuple Arc B.Q. The Sextuple Arc C.Q.

PREFACE TO PART FOURTH.

In the Appendix to Part Third, we have shown that the complementary arc of curvature to the octant is the arc of a circle described with a radius which equals 5·85223.. taking the radius of the primary octant as equalling 10. The immediate importance of the fact thus shown is in the independant demonstration which it furnishes of the actual arc-length of the octant.

In the same connection, as another additional analysis independently establishing the same result, we may refer to the application of the linear differential method in the analytical and synthetical examination of the line *J.K.N.*, which will be found at page 18 of this division of the work, (Part Fourth).

The quantitive relations of the octuple and sextuple arcs, as exhibited herein, we wish to particularly commend to the attention of the practical mathematician, not only in respect to the great utility which, as we opine, may be derived from the distinct knowledge and application of those relations, but also as an introduction to the systematic analytical investigation of the circle by divisional arcs and decimal circles.

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THE CIRCLE AND STRAIGHT LINE.

'Prove *all* things; hold fast *that* which is *good*.'
St. Paul.

PART FOURTH.

ANALYSIS OF THE CIRCLE.

Analytical Examination by Fig. 23.–(R.)

The Arc of 30 *degrees* $(\frac{C}{12})$. *The Octuple-arc B.Q.*[*] *(or B.K. of Fig.* 24.) *And the line B.b., which is a production of the chord of the octuple arc.*

The arc of 30 degrees B.e., [†] or D.e., has its sine equal to 5·0000.

If the Octuple arc B.Q. be bisected, and the secant to the half-arc be drawn, the line A.O.K.' so drawn shall intersect the line B.b. in the point bisecting the chord of the Octuple arc, and the line A.O.K.' shall be (manifestly) equal and similar to the line B.b. (Because B.b. bisects C.D., and A.O.K.' bisects B.D.)

---

[*] We have adopted this term for the moment to distinguish this arc of which the sine is to the tangent of the half-quadrant in the ratio of 8 to 10, of which the chord is to the tangent of the half-quadrant as 'the square-root of 80' to 10 and of which the tangent is to the tangent of the half-quadrant as 8 to 6. It is to be noted that the same (octuple) arc is the particular subject of the theorem and prop. belonging to Fig. 24, page 10, Part Third, and is further considered in the following supplementary examination by Fig. 24. (R.)

[†] B.e. of Fig. 31.—in which B.e. = V.e. = e.d. = C.d.

The secant A.O.K.' to the half-octuple arc, bisects the tangential line B.D., and consequently the tangent to the half-octuple arc is also equal to 5·0000. And equal to the sine of the arc of 30 degrees.

We have therefore the arc S.e., equal to half of the arc B.e., which is one third of the quadrant B.C.; and also the sine of B.e. equal to one half of B.D., the tangent to the half-quadrant, and equal to the tangent of half the octuple arc.

The Cosecant of the arc of 30 degrees equals 20·0000 (i.e., twice the radius);

The Tangent = 5·7735...
" Cosine = 8·66025...
" Cotangent = 17·3205...(i.e., twice the Cosine.)

A particular arithmetical relationship is at once apparent on contrasting these three quantities with each other... viz., the least is one third of the greatest, and the intermediate term is the one half of the greatest, moreover the least and the greatest of these numbers form with the number 10 a proportion of which 10 is the reciprocal or intermediate term. Now, 10 is the tangent to the half-quadrant, and the greatest of these numbers (17·32051) is the tangent to the arc of sixty degreee.

So that, herein we have a remarkable relation between a magnitudinal and arithmetical proportion. The inter-relation between the arcs contrasted with the inter-relation between the tangents is remarkable in respect to the dissimilarity in the correspondence (so to speak); for, in the arcs, the addition of one-half the first term gives the second, and the third term is twice the first (30°, 45° and 60°); but in the case of the tangent, cosine and cotangent of the arc of 30°, the third term is twice the second and three times the first.

5·7735, 8·66025, and 17·3205;

and in the tangents of the three arcs compared together, the second term is greater than the first in the same ratio that the third is greater than the second,

5·7735 : 10 :: 10 : 17·3205.

Scholium.—In considering the relation of the arc of 30° $(\frac{C}{12})$ to the half-quadrant $(\frac{C}{8})$, it should be noted that, if the chord of the arc of 30° be bisected, the one-half is the chord of an arc of 30° belonging to a circle of half the magnitude, (i.e., drawn on one-half the scale,) and is also the sine of the arc of 15° ($\frac{C}{24}$) cut off from the greater arc by bisection. Since the arc of 15° is one-third of the half quadrant this relationship is of especial interest.

The definite divisions of the lines obtained by this construction arc:—the line B.D. into three equal parts by perpendiculars drawn from the points *z.* and *y.* respectively. (Note.—If the tangent to the octuple arc be drawn, the difference, by which this tangent is greater than the tangent to the half-quadrant, is equal to one of the three equal divisions of B.D.)

The line B.D. divided into five equal parts by perpendiculars drawn from the points Q'.o.t.Q. respectively.

The line B.b. into fifths of which b.Q. is one fifth, Q.O. is two-fifths, and B.O. is two-fifths.

The secant A.D. into three equal parts by the points *z.* and *y.*

Note.—To assist the student in distinguishing the trigonometrical relation and values of these divisions, the following may be found useful. Figs. 1. and 2.—In each of these figures the line B.b. bisects the perpendicular *a.* C.; which line (*a.*C.) divides the triangle A.D.B into two similar and equal triangles. Fig. 3., (in connection with Fig. 23. (R.), indicates the harmony of the cyclometrical and trigonometrical interrelation of the lines and their equal divisions.

Fig. 26—This figure belongs to Fig. 25 *R.R.* 2, illustrating the decimal relation of the versed sine. (*See the commencement of Part Second.*)

For the purpose of examining and comparing the elements of a series of primary decimal circles, this form (Fig. 26) should accompany Fig. 25 *R.R.* 2, and Fig. 14. For purposes of comparative cyclometry, we deem this figure and its obvious modifications of great utility.

Fig. 24. (*R.*)—*The Octuple-Arc B.Q.* $\left\{\begin{array}{r}\text{Sine}=8\cdot0000.\\ \text{Co-sine}=6\cdot0000.\end{array}\right.$

B.K.', the tangent to the half-octuple-arc...
B.K.' : B.D. : : 5 : 10.

B.Q. the chord of the octuple-arc...
B.Q. : B.D. : : $\sqrt{80}$: 10.*

B.Y. equals the sine of the octuple-arc...
B.Y. : B.D. : : 8 : 10.

B.E. the tangent of the octuple-arc...
B.E. : B.D. : : 8 : 6.
(Because B.E. = (B.D. + B.H.')
B.E. = 13·3333...&c.—

A.P. the cosine of the octuple-arc...
A.P. : B.D. : : 6 : 10.

A.E. the secant of the octuple-arc...
A.E. : B.D. : : 10 : 6. And A.E. $=\dfrac{100}{6}$

Since B.Q'.z.k. bisects A.C. and B.O.y.b. bisects C.D., the part Q.'Z. of the octuple-arc D.Q.', is equal to the part O.Z. of the same arc. Now the point O. is the point of bisection of the chord of the (respondent) octuple-arc B.Q., and therefore if the octuple-arc B.Q. be described of one-half the magnitude, B.O. will be the chord thereof. But D.O. is the sextuple arc of the quadrant D.A. and responds to the sextuple-arc B.t. of the quadrant B.C.; consequently the point at the extremity of the octuple-arc of half-magnitude coincides with the point at the extremity of the responding sextuple-arc.

Examination by Figs. 23. (R.), and 24. (R.)

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* Hence the sine is to the chord of the octuple arc : : 8 : $\sqrt{80}$. Also—the sine of the octuple arc is to the sine of the half-quadrant : : 8 : $\sqrt{50}$. And the chord of the octuple arc is to the chord of the half-quadrant : : $\sqrt{80}$: the square-root of 'twice the versed sine of the half-quadrant multiplied by 10.' (i.e. $\sqrt{58\cdot5786.}$)

The Sextuple-Arc—B.t. *Sine* = 6·0000.

The close relation of this arc to the octuple-arc of which the sine = 8·0000, will be made at once apparent by considering C.D. as the tangent and by taking the point C. for the original instead of the terminal extremity of the quadrant, in which case the point t. becomes the extremity of the octuple-arc C.t.*

The following are the comparative elements of the two arcs. The radius equalling 10.

	Octuple-arc.	*Sextuple-arc.*
The Sine =	8·00000. .	6·00000 . .
" Chord † =	8·94427. .	6·32455 . .
" Tangent =	13·3333, &c.	7·50000 . .
" Secant =	16·6666, &c.	12·50000 . .
" Co-sine =	6·00000. .	8·00000 . .

These two arcs (the Octuple and Sextuple) are also so related to the half-quadrant B.S. that the one is less by the same quantity by which the other is greater than the half-quadrant, i.e.,—the point at the terminal extremity of the half-quadrant is at an equal distance between the points at the terminal extremities of the two arcs. The relationship, therefore, is :—

B.S. + half the diff. of B.Q. and B.t. = B.Q.
B.S. − half the diff. of B.Q. and B.t. = B.t.
B.Q. + B.t. = 2 B.S. . . . . = B.C.

From this relation the quantitive values of the two arcs are readily obtainable and we find accordingly :—

$$\left.\begin{array}{l}\text{The Octuple-arc contains } 53\tfrac{1}{8} \text{ degrees}\\ \text{The Sextuple-arc contains } 36\tfrac{7}{8} \text{ degrees}\end{array}\right\} 45 \times 2$$

Together equalling 90 degrees. = B.C.

Note—For the purpose of assisting the examination and appreciation of the linear divisions, we furnish the following: Figs. 4. 5. 6.—In Fig. 4, the arc B.g. is described

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[*] Evidently therefore the sextuple and the octuple arcs are so related that the one is the complement of the other; it is none the less important to clearly distinguish between them.

[†] Hence the chord of the octuple-arc is to the chord of the sextuple-arc : : $\sqrt{80} : \sqrt{40}$.

with the radius A.B., and the arc B.p. with the radius C.B. which is the one-half of A.B. The tangent B.D. of the arc B.g. is divided into four equal parts in the points a.b.d. and the lines drawn from these points respectively to the central point A. divide each of the arcs B.g. and B.p. into four (unequal) parts. It will be found that the ratios of the divisions of these arcs to each other (i.e. of each to the next) are proportionals; thus—g.f. : f.e. : : f.e. : e.c. : : e.c : c.B. Consequently, f.c. − g.f. = c.B. (or f.c. − c.B. = g.f.), and g.f. + c.B. = f.c. That is the arc-length of g.f., together with the arc-length of c.B. equals half the length of the arc B.g. Now if the radius be reduced to the one-half and the arc B.p. described therewith be examined, the ratios of the arcs will be found proportionals as before, and we have p.o. : o.n. :: o.n. : n.m. :: n. m. : m.B.; therefore the arc-lengths p.o. + m.B. = o.m., and o.m. equals the half of B.p. The Figs. 5. & 6. are, in the first place, for the purpose of showing that the same inter-relation of the divisions of an arc divided in the same manner – to wit, by division of the tangent into equal parts—holds good whether the length of the radius be increased and the extent of the arc be diminished, or whether the length of the radius be diminished and the extent of the arc be increased. The same figures serve to illustrate the trigonometrical relation of the radial lines drawn from A., which intersect the tangential line B.D. The perpendiculars at the points d. b. a. divide the line A.D. into four equal parts, and the line A.d. into three equal parts, and the line A. b. into two equal parts. The perpendiculars themselves are in arithmetical progression and are successive multiples of the least. Consequently d. + B. = b. + a.

Figs. 7. and 8. are to illustrate the conditions which necessitate the proportional inter-relations of the divisional arcs; also, in Fig. 7., we have a series of similar triangles, of which the ratios are proportionals A.c.d. : A.d.B. : : A.d.B. : A.B.D. : : A.B.D. : A.D.f.; the bases of these triangles are therefore similarly proportional, each to the next. The two sides D.e. and e.B. of the triangle D.e.B. are together, a mean proportional between D.B. and g.f., and, therefore, g.f. is greater than D.f. in the same proportion that D.e.B. is greater than D.B., &c.

Fig 4

Analytical Illustrations.

Figs. 4, 5, 6, 7, 8.

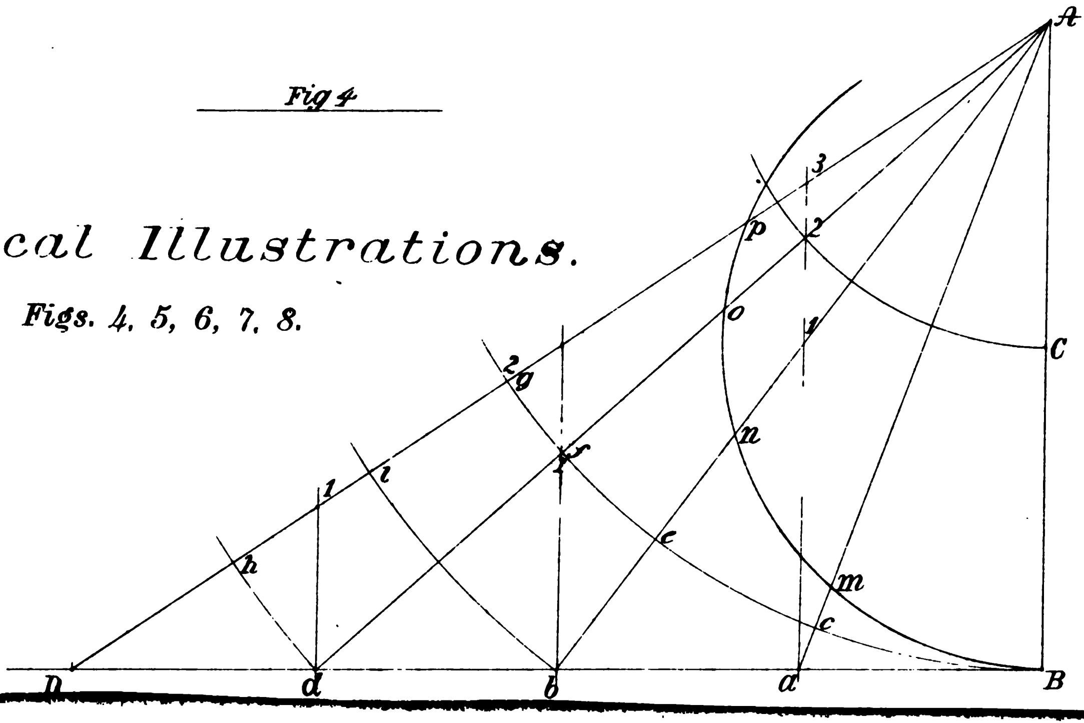

Fig. 31.—*Examination of the figure.*

Of the quadrant B.C....Q.C. is the complement of the octuple-are B.Q....B.d. (two thirds of the quadrant) = C.e.; ... C.d. = d.e. = e.B.; and C.d. + d.e. + e.B. = C.d. × 3 = C.B.... S.d. = S.e., and S.Q. = S.P.; therefore Q.d. = P.e.

Of the similar quadrant respondent through S.d., to wit, the quadrant (C.)(B.) the similar divisions are equal; thus (C.)(d.) + (d.)(e.) = C.d. + d.e.; also V.(e.) = B.e. = (e.)(d.) = (d.)(C.), &c.

Of the similar quadrant respondent through e., to wit, the quadrant A.D., the similar divisions are equal; thus Q'.Z. is equal to Q.S., Q'.e. is equal to Q.e., and the line K.K.' bisects the line joining Q.Q'.

Of the arcs shown in this Fig. the arc of curvature described with radius b.𝒳. intersects the central point K. of the square, which is also intersected by the arc of the sine length described with radius B.W.

We note, in the first place, the remarkable relation between the square and the quadrant herein exhibited, and which may be thus stated as a theorem... that, if a quadrant be described in a square, and if the two adjacent sides of the square next the quadrant be bisected and a square half the magnitude of the first be formed by joining the points of bisection and the central point of the greater square, then will the quadrant be intersected by the two sides of the lesser square, and the two points of intersection shall divide the quadrant into three equal parts. This theorem may be said to be demonstrated by inspection of the figure—that is, in other words, it becomes manifest—but, moreover, we have the central part of the three into which the quadrant is divided, bisected by the line K.S.D. (half the secant of the half quadrant), and we are familiar with the quantitive values of the lines K.S. and S.D., of which K.S. equals the versed sine of the half-quadrant = 2·92893... and S.D., the difference of the radius and secant of the half-quadrant, = 4·14214... (i.e., taking the side of the greater square equal to 10·0000.)

We may therefore, by taking the distance K.d. as a radius, describe a quadrant terminated by the points d. and e. which terminate the arc d.e.; or, by taking the line K. S. as a radius we may describe an arc touching the arc cut off from the greater quadrant, at the central point thereof S.; and the proportion of these arcs each to each and their quantitive values may be very readily determined.

Now if a fourth quadrant intermediate between the two last be described by making the radius equal to one-third of the radius A.B., this last quadrant will intersect the central part of the primary quadrant in two points, and if the point (K.), belonging to the primary responding quadrant (C.)(B.), be taken as a centre and with the same radius—to wit, equalling one-third of A.B.—another secondary quadrant be described, we then obtain Fig. 32.

Fig. 32.—In this figure the line 𝔛.𝔛. occupies the half distance between the perpendicular diameters of the two (greater) circles, and since the radius of each equals the one-third of ten (3333...) those circles intersect each other on the line 𝔛.𝔛. Setting aside for the moment the preceding demonstrations of the locality and characteristics of the line 𝔛.𝔛, it is at once apparent that the relative place of that line as shown in the figure necessarily belongs to the structural plan of the circle, because K.S. equals S.T. or (S.W.) and the distance S. 𝔛. is included in the distance S. T. and the distance, also, between the lines (T.) V. and K.k'. is included in the side of the right-angled triangle of which K.S. is the base, therefore if S. be taken as the centre of a circle described with the radius S.K., that circle will intercept the point T., and if the point (S.), responding to S., be taken as the centre and a circle be described with the radius (S.)(K.), that circle will intercept the point (T.) The distance between the centres K. and (K.) of the two (greater) circles is therefore necessitated and determined by the actual relation of the lines. Now taking our demonstration of the quantitive value of R.𝔛., as the arc-length of the half quadrant—to wit, 7·85674.. — we have 𝔛.T. = (R.T.—R.𝔛.) = 2·14326...But S.𝔛., by that demonstration, equals the one-tenth of R.𝔛., i.e., 785674..., and S. K., by the figure, equals S.T., and equals S. W. the versed sine of the half quadrant and of which the magnitudinal value is therefore 2·92893... *Let the fact be particularly noted that* 2·14326 + ·785674 = 2·92893.

It may be observed that, since the chord of the arc of sixty degrees equals the radius, several of the distances between the points indicated in this figure are necessarily equal each to each.

Fig. 33.—If the responding quadrant (B.)(C.) be located at a distance from the centre K. of the primary circle, a little greater than its true place as determined by the arc-length of the half quadrant—to wit, by making the distance R.𝒳. equal 7·86565. . instead of 7·85674...the perpendicular W.S., produced through S., will intercept the responding quadrant (B.)(C.) in the point d' (on the line J. K.), and in that case d.'K. will be equal to d'.S., and d.d'. equal to (S.)S. And also, if from K. a perpendicular be drawn through the line R.T., at E., to the point e. in the quadrant B.C., the perpendicular K.e. is manifestly equal to K.d., and K.E. equal to K.d'. Hence we obtain the formation of four squares equal each to each, and of eight parallelograms equal and similar each to each, as shown in Fig. 33.

Of these squares, each of the sides is equal to K.d'. and the diagonal equal to K.S.... Now the quantitive value of K.S. is 2·92893 ... and K.d'. : K.S. :: 10 : 14·14214... therefore K.d.' = 2·07107; but in place of a diagonal each of the parallelograms, of which the two sides are each equal to K.d.' and the other two sides each equal to d.'d., or S. (S.), contains a fraction of the quadrant B.C., and which fraction is the one-sixth part of the quadrant, (an arc therefore containing 15 degrees described with radius equal to 10.)

Analytical Examination by Fig 31.

The arc of 15 *degrees* $\left(\frac{C}{24}\right)$; *and the line J.K.N.*

The arc of 30° B.e. is bisected in the point m.; B.m. is therefore an arc of 15°. B.m. = m.e. = e.S. B.m. + m.e. + e.S. = B.S. S.d. is also an arc of 15° and equals S.e. = B.m.

Since the line A.M.m. bisects the angle B.A.f., the line C.d.g. similarly bisects the responding angle D.C.F. and the point M. manifestly responds to the point d., therefore the line C.d. bisects an are of 30° and J.d., on the line J.K., is the tangent to an arc of 15° described with the radius C.J. which equals one half the radius A.B., and J.d. therefore equals one-half of B.n. the tangent to the arc B.m.

But the lines A.d. and A.e. are likewise respondent each to the other, and, therefore, the point d., at the terminal extremity of the arc C.d. responds to the point e. at the terminal extremity of the arc B.e. Now the sine of the arc B.e. = (the half of B.D.) = 5. And consequently the cosine of the arc B.e. equals 8·66025. Therefore, N.d., which responds to the cosine of the arc B.e., also equals 8·66025. . . and J.d. = (10 − 8·66025) = 1·33975. . . Again N.M., at the opposite extremity of the line, responds to J.d. and also equals 1·33975. . . ; and N.L. : the sine of B.e. : : 2·88675 : 5.* Therefore, M.L., which is the distance between the points in which the line is intersected by the arc and by the chord of the arc respectively, equals (2·88675 − 1·33975) = 1·54700.

The magnitudinal values of these distances having thus been with certainty ascertained, the entire line J.K.N. may be analytically examined; commencing from the extremity N. we have :—

N.M.	=	1·33975...	
M:L.	=	1·54700...	
L.K.	=	2·11325...	
N.K.—		=	5·00000
K.d.'	=	2·05323...	2·07107
d.' $	=	·01784...	
$ y.	=	·04218...	0·78567
y. 𝔛.	=	·74349...	
K.𝔛.		=	7·85674
𝔛.y'	=	·74349...	
y.' ($)	=	·04218...	
($) d.	=	·01784...	
d.J.	=	1·33975...	
𝔛.J.			2·14326...
N.K.J.		=	10·00000

Or again, by taking the (central) point 𝔛., we have as definite parts of the line J.K. :—

𝔛.y. and 𝔛.y.' each of which equals ·74349...
𝔛.$. and 𝔛.($).' each of which = .785674...
𝔛.d. and 𝔛.d.' each of which = ·80351...

and finally

𝔛.T = 2·14326, and 𝔛.K. = 2·85674.

---

* Because N.L. is the sine of the same angle, and 288675 : 5 :: 5 : 866025.

The definite division of the line J.K.N. thus determined evidently furnishes the means of again testing the correctness of that quantity of length which has been hitherto supposed to measure the half-quadrant, namely, ·78539... It appears almost needless to show here by the addition and subtraction of the figures that such quantity cannot be made to harmonize with those definite divisions of the line J.K.N., which directly result from trigonometrical measurement, and that, furthermore, no quantity, as representing the arc-length of the half-quadrant, can be made to harmonize therewith other than the quantity 7·85674..

As an example, let us first take this demonstrated quantity;—to wit, N.𝒳. = 7·85674:—

We have :— 'J.K = 5·00000
K.𝒳. = 2·85674 } =
𝒳.d. = ·80351 } = 3·66025

J.d. = 1·33975

And 2 J.d. = 2·67950, the tangent to the arc of 15° (B.n.)
But assume N.𝒳. = 7·8539...
Then :— J.K. = 5·0000
K.𝒳. = 2·8539... }
$\mathscr{X}\text{.d.} = (\mathscr{X}.\$+\$\text{d}) = \cdot 8007\ldots\ \} = 3\cdot 6546$

J.d. = 1·3454

And 2 J.d. = (2·6908), as the tangent to $\left(\frac{C}{24}\right)$ the arc B.n.

(Note.—We have already shown that assigning 7·8539... as the arc-length of the half-quadrant, is in fact attributing two different lengths to the same line. See Part Second.)

Noteworthy in Fig. 31, is the foursided figure d.e. M.H. of which each of the sides is an arc of 30°, containing the central one-third of a quadrant described with radius = 10. Of the two longer diameters H.e. and M.d. each equals 7·3205... ; of the two shorter diameters S.U. and z.u. each equals 5·85786...

Fig. 34.—If the lines A.H. and B.e. be produced through the points H. and e. respectively, until they meet each other they will meet at a distance from the point d. equal to the

radius A.B. *(or from the point J. equal to the distance Nd.) and if from the point where the two lines so meet as a centre an arc A.B. be described, the length of the radius = 18·6025... and the chord of the arc = 10.†

THE ARC OF $18\frac{7}{16}$ DEGREES. (*The half-sextuple arc.*)

Fig. 34.—Bisect the arc B.P. (i.e., the sextuple arc) ; and through the point of bisection c. draw the line A.Q.m.

The arc B.c. is therefore an arc containing $18\frac{7}{16}$ degrees ‡ its linear elements are,

The Sine........ = 3.16227...
" Co-sine..... = 9·48685...
" Tangent... = 3·33333... &c.
" Co-tangent = 30·0000...
" Secant...... = 10·5409...
" Co-secant.. = 31.6227...

(Note.)—*These figures may be compared with the elements of the arc of* 18°,

The Sine........ = 3·09017...
" Co-sine.... = 9·51057...
" Tangent.. = 3·24920...
" Co-tangent. = 30·7768...
" Secant.... = 10·51462...
" Co-secant. = 32.36068...

Since the secant of the arc B.c. intersects the point Q., at the extremity of the octuple arc, D.Q., the quantitive magnitudinal values of the lines belonging to the arc may be very readily computed and verified ; the one-third of ten as the tangent to the arc presents itself at once... thus: —the triangle A.Q.O. is a part of the similar greater triangle A.m.B. ;

---

* d.A. also equals A.B. .

† Having regard to the relations of the triangle thus formed the base of which (i.e. the chord of A.B.) equals 10, it is probable that the characteristics of this triangle will render it of much utility if applied in the art of computation.

‡ The sextuple arc has been shown to contain $36\frac{7}{8}$ degrees. (Page 13.)

Now we have A.O. = 6, and Q.O. = 2; *
(As A.O. : Q.O. :: A.B. : m.B.)

Therefore as 6 : 2 :: 10 : $\frac{10}{3}$ which represents m.B. the tangent to the arc.

We have confined these illustrations thus far to the lines and divisional arcs belonging to a circle described with a radius equal to 10. It is evident that by taking any one of these primary or more important lines as a radius, a series of quantitive magnitudes with similar inter-relations will be obtained, and which will have, through the radius common to all of them, a definite and known relationship to the lines and divisional parts of the primary circle. Since some of these secondary lines may absolutely agree in magnitude with some of the primary lines or may have some very simple (quantitive and numerical) relationship to them it is very desirable that the relations should be investigated and classified. As a brief example we will take the line 5·85786... which equals twice the versed sine of the half quadrant and has been now shown to be one of the most important of the primary lines belonging to the half-quadrant.

If from the point B., Fig. 34. on the line B.A., a centre $\bar{a}$. be taken at a distance = 5.85786... and from the centre $\bar{a}$. with a radius $\bar{a}$.B. a quadrant B.c. be described, then if the quadrant be bisected in the point h., B.h. shall be the half-quadrant; and if one third of the quadrant be divided off at the point $\bar{d}$. the remainder—to wit, the arc B.$\bar{d}$. – shall be an arc containing 60°. and the versed sine of this secondary arc of 60°.(described with D.V.S.radius $\bar{a}$.B.=5·85786), shall equal the versed sine of the half-quadrant belonging to the primary circle, (because the versed sine of the arc of 60°. equals one-half its radius).

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* Geometrical demonstration that the sine of the octuple arc is to the tangent of the half-quadrant as 8 : 10 will be found at page 10, Part Third, Fig. 24.

Taking, therefore, as the example, the half-quadrant B.h. we have :—

Radius = 5·85786.

The half-quadrant. $\left(\frac{C}{8}\right)$

The Sine...... = 4·142134...
" Tangent. = 5·85786...
" Secant... = 8·28426..

Herein we have an agreement (coincidence) between certain of the principal lines belonging to one circle and certain of the principal lines belonging to another circle differing from the first in magnitude; the lines belonging to the second being dissimilar* from the lines belonging to the first, with which they agree in length.

The decimal Circle of the D. V.S.†

The arc of sixty degree. $\left(\frac{C}{6}\right)$

The Sine...... = 5·0730... ‡	The Co-sine...... = 2·92893
" Tangent =10·1461...	" Co-tangent. = 3·38204
" Secant.. =11·7157...	" Co-secant.. = 6·76408
" Chord.. = 5·8578.	

(Note.)—The chords of the secondary arcs coincide with and form a part of the chord of the primary arc if the arcs be so described that each commences from the same original point (B.) as that of the primary and agrees with the primary arc in position, which is the method adopted in the foregoing illustrations: Since the primary radius includes the radius of each of the circles, the extremity of the sine may be made the point of coincidence, as in some of the analytical figures of our Part Second, (See Fig. 12.); Or, again, the same point may be taken throughout for the centres of the circles—that is, the arcs may be described all from the one centre—in which case the secant will be the line of coincidence. In systematic analysis it will evidently be convenient to adhere to one uniform method throughout the whole, or, at least, throughout a part, of the figures belonging to the series.

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* That is, they are constructively different in relationship.

† That is, the duplicated versed sine of the primary half-quadrant.

‡ Sine 5·073056670......taking $\sqrt{50}$ = 7·07107.

Analysis by Decimal Circles.

Illustrated in Figs. 35 (a.) and 35 (b.)

In which Figures these different methods are applied to illustrate the system of decimal cyclometry.

In Fig. 35 (a.) the tangent is made the line of coincidence; in Fig. 35 (b.) the sine is the line of coincidence with reference to the radius A.B., and also the secant is indicated as the line of coincidence with reference to the radius A.C., and illustrated by the arcs intersecting that line.

The decimal system of cyclometry is illustrated in these two figures by the three decimal circles, namely, the circle of the sine; the circle of the duplicated versed sine; and the circle of the secant. These lines may be termed 'Capitals' of the system; they are all secondary to the radius A.B. which equals 10 and is the 'primary' of the system; but each of them is 'primary' to the lines and divisional arcs belonging to the circle of which it is the radius, for instance, in the example which has just been given, the circle of the D. V. S. has for its 'primary' the line A.B. which is the radius of the primary quadrant, but the line $\overline{\underline{a.}}$ B. (equalling twice the versed sine of the primary quadrant) is the primary to the half-quadrant, to the arc of 60°, and to all other divisional arcs, belonging to its own circle, and also primary to the lines belonging to those respective arcs. In constructing a cyclometrical table it is evident that this system may be pursued by sub-divisions as far as may be found desirable. The immediate gain to the science of quantity and number from such a tabulated analysis of the circle would be the increased knowledge obtained of the interrelations of the subjects of that division of science; for example, by taking a primary radius equal to 10, we find that the number 5 represents the sine of the arc of 30 degrees; the number 8 represents the sine of the octuple arc; and the number 6 the sine of the sextuple arc; we become thereby aware of and are able to appreciate a particular magnitudinal relationship between these numbers, and not only of each to each of these but also of each and of all of these to many other (magnitudinal quantities) numbers. Or, as another example, we may take again the number 5, which appears as the sine of the arc of 30° and the cosine of the arc of 60°, both belonging to the primary circle, and also appears as the sine and cosine of the half-quadrant belonging to the decimal-circle of the sine—namely, that which has the sine-length $\sqrt{50}$ as its (radius) 'Capital.'

Examination of these two plates, 35 (a.) and 35 (b.) together with the annexed table will suffice to render the cyclometrical decimal system clearly understood.

NOTE.—In Fig. 35 (b.), wherein the line of coincidence includes the sines of all the arcs, the division of the primary radius A.B. should be noted. The entire part divided off above the line of coincidence is equal to the sine-length of the primary (7·07107); from centre A., at the upper extremity of A.B., a part is divided off by the centre of the dec-circle D. V. S. equal in length to the versed sine of the primary; and, adjoining the line of coincidence—above that line—a part is divided off by the centre of the circle of the secant, also equal in length to the versed sine of the primary half-quadrant.

The method of analysis to which these figures belong will be better appreciated after consideration of Table II., which we furnish as an appendix, and in which the arc-lengths are included as elements of the circles. From that table it will immediately appear, for example, that the arc-length of the arc of nine degrees belonging to the decimal circle of the sine equals the radius A.B. divided by nine; and that the arc-length of the half-quadrant belonging to the same circle equals five times that quantity.

In concluding these illustrations we beg to state that they may be considered generally as contributions and suggestions towards an analytical investigation of the circle; their immediate purpose however is to illustrate the proposition that the circle itself is not only a reality but is one of the primary facts of Creation, constituting, indeed, a central primary or great fundamental fact in or upon which very many of the secondary facts belonging to abstract science † have their common basis.

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† Abstract-Science may be defined as that division of Science which treats of the inter-relation of like subjects of Science, in respect to those general properties (number, quantity, condition, form, magnitude,) which belong to those subjects as existences (things).

CYCLOMETRICAL TABLE I.

Decimal Series. *Primary radius A.B.* = 10.

(1) Divisional Arcs of the primary.	Half-quadrant.	Arc of 60 deg.
Sine	7·07107	8·66025
Tangent	10·00000	17·32051
Secant	14·14214	20·00000
Co-sine	7·07107	5·00000
Chord	7·65366	10·00000
Co-tangent	10·00000	5.77350
Co-secant	14·14214	11.54701

	Arc of 30°.	Arc of 15°.
Sine	5·00000	2·58819
Tangent	7·77350	2.67949
Secant	11·54701	10·35275
Co-sine	8·66025	9·65926
Chord	5·17638	2·61052
Co-tangent	17·32051	37.32050
Co-secant	20·00000	38·63702

	Octuple Arc.	Sextuple Arc.
Sine	8·00000	6·0000
Tangent	13·33333	7·5000
Secant	16·66666	12·5000
Co-sine	6·00000	8·0000
Co-secant	12·50000	16·6666
Co-tangent	7·50000	13.3333
Chord	8·94427	6·32455

(2) Decimal Circle of the Sine. Radius = 7·07107.	Half-quadrant.	Arc of 60 degrees.
Sine	5.00000	6.12372
Tangent	7·07107	12·24744
Secant	10·00000	14·14213
Co-sine	5·00000	3·53553
Chord	5·41195	7·07106
Co-tangent	7·07107	4·08248
Co-secant	10·00000	8·16496

	Arc of 30°.	Arc of 15°.
Sine	3·53553	1·830127
Tangent	4·08248	1·894687
Secant	8·16497	7·320506
Co-sine	6·12372	6·839313
Chord	3·66025	1·845919
Co-tangent	12·24745	26·389589
Co-secant	14·14213	27·321252

	Octuple Arc.	Sextuple Arc.
Sine	5·656856	4·242642
Tangent	9·428093	5·303302
Secant	11·785116	8·838837
Co-sine	4·242642	5·656856
Co-secant	8·838837	11·785116
Co-tangent	5·303302	9·428093
Chord	6·324555	4·472133

(2) Decimal Circle of the D. V. S :* Radius = 5·85786.	Half-quadrant.	Arc of 60°.[1]
Sine	4·14213	5·073053
Tangent	5·85786	10·146106
Secant	8·28426	11·715728
Co-sine	4·142134	2·928932
Chord	4·490456	5·857864
Co-tangent	5·857864	3·382040
Co-secant	8·284268	6·764080

	Arc of 30°.	Arc of 15°.
Sine	2·928932	1·511121
Tangent	3·382040	1·569609
Secant	6·764080	6·044482
Co-sine	5·573060	5·659561
Chord	3·032252	1·529207
Co-tangent	11·146120	21·794185
Co-secant	11·715728	22·638244

That is, of the duplicated versed-sine.

	Octuple Arc.	Sextuple Arc.
Sine	4·68628	3·514716
Tangent	7·81048	4·393395
Secant	9·76310	7·322325
Co-sine	3·51471	4·686288
Chord	5·23942	3·70841...
Co-tangent	4·39339	7·81048...
Co-secant	7·32232	9·76313...

(4) Decimal Circle of the Secant * Radius = 4·14214	Half-quadrant.	Arc of 60°.
Sine	2·92893	3·587196
Tangent	4·14214	7·174392
Secant	5·85787	8·284274
Co-sine	2·92893	2·071069
Chord	3·17025	4·142138
Co-tangent	4·14214	2·391466
Co-secant	5·85787	4·782932

	Arc. of 30°.	Arc of 15°.
Sine	2·071068	1·096936
Tangent	2·391466	1·109883
Secant	4·782932	4·439532
Co-sine	3·587199	4·001002
Chord	2·144129	1·081315
Co-tangent	7·174398	15·458674
Co-seant	8·284272	16.004008

	Octuple Arc.	Sextuple Arc.
Sine	3·313712	2·485284
Tangent	5·522853	3·106605
Secant	6·903566	5·177675
Co-sine	2·485284	3·313712
Chord	3·704841	2·61971
Co-tangent	3·106605	5·52285
Co-secant	5·177675	6·90356

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[*] We thus apply the term 'Secant' to the difference of the Secant and Radius of half-quadrant. Since the Secant is twice the Sine there is no danger of misapprehension from such application; some other term might, however, be preferable.

APPENDIX.

We have endeavoured as much as possible to avoid the unnecessary use of strange and unusual terms. It might be preferable instead of the expressions 'decimal system' and 'decimal circles' to substitute the word 'decuple' for 'decimal' in order to express the relationship, and to write 'decuple circles' belonging to the 'cyclometrical decuple' system' A decimal system in accordance with the more customary sense of the expression would be formed by taking as the successive 'capitals' of the system the numbers 9, 8, 7, 6, 5, 4, 3, 2, 1; that is, the radius of the first circle, to be nine-tenths the length of the 'primary,' of the second circle eight-tenths of the primary, and so on. A quadratic decimal system might also be formed by adopting the same 'primary' (*viz.* the number ten) as the square root of 100, the first capital would then be the square root of 90, the second capital the square root of 80, and so on. In a system so framed the circle of the sine would find its place with its capital as the square root of 50. For the present, however, it appears more desirable to extend the table now given by addition of other important divisional and related sub-circles; as a first addition we may take the half-octuple, the half-sextuple, the half-octant * and the arc of nine degrees; we have accordingly :—

CYCLOMETRICAL TABLE II.

The Primary.—*Radius* = 10. *Arc-length of Octant* = 7·85674

	Half-Octuple $26\frac{9}{16}$ deg.	Half-Sextuple $18\frac{7}{16}$ deg.	Half-Octant $22\frac{1}{2}°$	$\frac{C}{40}\left\{\begin{array}{c}\text{Arc}\\ \text{of } 9°\end{array}\right\}$
Arc..........	4·63766	3.219081	3·92837	1·571348
Sine....... ..	4·47213	3·162277	3·82683	1·56434
Tangent....	5·00000	3·333333	4·14214	1·58384
Chord.	4·59507	3·203728	3·90179	1·56917
Secant......	11·18033	10·540929	10·82394	10·12465
Co-sine	8·94427	9·486833	9·23881	9·87688
Co-tangent.	20·00000	30·000000	24·14213	63·1375
Co-secant..	22.36066	31·622787	26·13128	63·9245

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* The 'Octant' is a more convenient expression than 'half-quadrant,' and, in like manner, the term 'Sextant' commends itself for general use to denote the arc of 60 degrees.

Dec-Circle of Sine—Radius = 7·07107 *Arc-length of Octant* = 5·55555

	Half-Octuple.	Half-Sextuple.	Half-Octant.	$\frac{C}{40}\left\{\begin{matrix}\text{Arc}\\ \text{of } 9^\circ\end{matrix}\right\}$
Arc............	3·27932	2·76234	2·77777	1·111111
Sine	3·16227	2·236068	2·70597	1·106155
Tangent......	3.53553	2.37023	2.92893	1·119944
Chord.........	3·24920	2·265379	2·75898	1·109576
Secant........	7·91468	7·453565	7·65368	7·260210
Co-sine	6·32455	6·708206	6·53282	6·984010
Co-tangent..	14·14213	21·21321	17·07107	44·64487
Co-secant	15·81140	22·36068	18·49467	45·20146

Dec-Circle of D.V.S.—Radius = 5·857854... *Arc-length of Octant* = 4·60237...

	Half-Octuple.	Half-Sextuple.	Half-Octant.	$\frac{C}{40}\left\{\begin{matrix}\text{Arc}\\ \text{of } 9^\circ\end{matrix}\right\}$
Arc............	2·716676	1·885689	2·301180	0·920492
Sine	2·619711	1·852417	2·241703	0·916368
Tangent......	2.928930	1·952621	2·426407	0·927781
Chord.........	2·691728	1·876683	2·285613	0·919198
Secant	6·549278	6·174666	6·340512	5·930816
Co-sine	5·239428	5·557212	5·411961	5·785738
Co-tangent..	11·715720	17·573580	14·122213	36·985063
Co-secant	13·078561	18·524182	15·321473	37·446076

Dec-Circle of Secant-—Radius = 4·14214... *Arc-length of Octant* = 3·245371...

	Half-Octuple.	Half-Sextuple.	Half-Octant.	$\frac{C}{40}\left\{\begin{matrix}\text{Arc}\\ \text{of } 9^\circ\end{matrix}\right\}$
Arc	1·920983	1·333388	1·627185	0·650874
Sine	1·852418	1·309856	1·585164	0·647971
Tangent......	2·071068	1·380713	1·715732	0·656048
Chord	1·903342	1·327029	1·616176	0·649975
Secant	4·631049	4·366201	4·483427	4·193751
Co-sine	3·704842	3·929579	3·826844	4·091143
Co-tangent .	8·284376	12·426414	10·000000	26·152436
Co-secant ...	9·262098	13·098599	10·823942	26·478422

With respect to the two tables here given we have not the slightest doubt whatever as to their substantial correctness; it is, however, quite possible that some arithmetical mistakes in computation may be found in them, and as to the last decimal places the figures of many of the quantities cannot be strictly accurate because those quantities are based upon the primary quantity 7·07107 as the square root of 50, and such is not exactly the square root which is 7·0710678... It is quite proper that the correction should be made and the figures be furnished with strict accuracy to the last decimal place, but the published tables of natural sines, &c., now authorized, or, at least, those most in use for general reference, are based on this 7.07107 and it is desirable in the first instance to show wherein our results are in perfect agreement with the results of recognized trigonometrical processes. The corrections may be very easily made and more complete tables furnished; as an example of the necessity of strict accuracy, worthy of note, we will specify the first of the elements of the half-sextuple arc belonging to the dec-circle of the secant, Table II., to wit, the arc-length thereof—which appears as 1·333388.. It is most probable that strict accuracy will give the figure 3 as an interminable decimal... that is, will give the actual quantity as $10 + \frac{1}{3}$. (By correcting the equivalent of the capital (radius) in the fifth decimal place the figure 8 becomes reduced to 5 in the fifth decimal place of the example, and, moreover, the half-sextuple element of the primary, from which the corresponding element of the dec-circle is derived, is very slightly in excess from the same cause).

Fig. 36.—Is to some extent a development of Fig. 31; it has, moreover, for its object to bring prominently under observation certain triangles and squares which, in their relation each to each and to the divisional arcs and elements of the circle, may be found to possess much utility.

The Primary Arc of the Radius.

We have omitted to include in the preceding tables, an arc which may be considered to belong especially to the primary series. It may be termed the primary arc of the radius and is the arc which in longitudinal magnitude equals the radius.

This arc doubled will, of course, equal the diameter: half the arc will equal the sine of the arc of 30 deg. and so on—thus :

The primary arc of the radius (equal in length to the radius.) } = 57·2754 . . . deg.

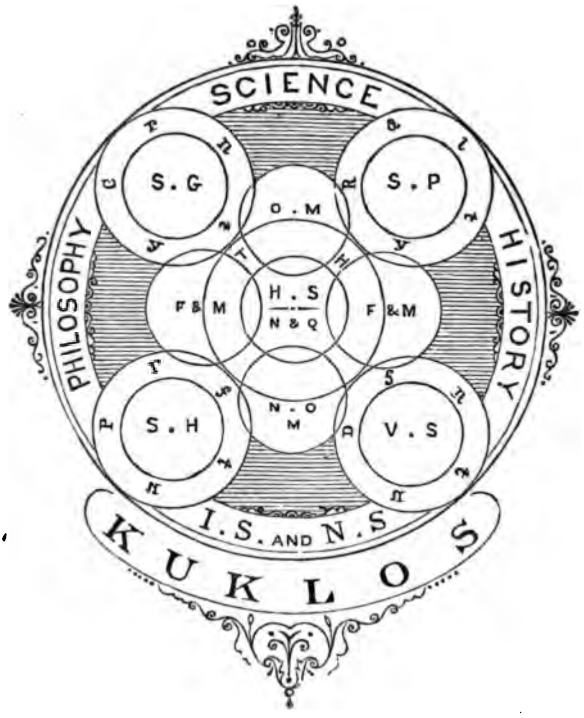
The primary arc of the diameter (equal in length to the diameter.) } = 114·5512 . . . deg.

The elements of these arcs should be added to the tables.

NOTE.—The arc-lengths belonging to the first cyclometrical table are as follows :—

(1) The Primary.	Half-quadrant	7·85674
	Arc of 60 degrees	10·47566
	" " 30°	5·23783
	" " 15°	2·61891
	Octuple arc – 53⅛°	9·27532
	Sextuple arc – 36⅞°	6·43816
(2) Dec-Circle of Sine.	Half-quadrant	5.55555...
	Arc of 60 degrees	7·407407...
	" " 30°	3·703703...
	" " 15°	1·851851..
	Octuple arc – 53⅛°	6·558642
	Sextuple arc – 36⅞°	4.552468
(3) Dec-Circle of D.V.S.	Half-quadrant	4·60237
	Arc of 60 degrees	6.13650
	" " 30°	3.06825
	" " 15°	1·53412
	Octuple arc – 53⅛°	5.43337
	Sextuple arc – 36⅞°	3·77137
(4) Dec-Circle of Secant.	Half-quadrant	3·25437
	Arc of 60 degrees	4·33916
	" " 30°	2.16958
	" " 15°	1·08479
	Octuple arc – 53⅛°	3·84197
	Sextuple arc – 36⅞°	2·66677

END OF THE APPENDIX TO PART FOURTH.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